Twenty-Sixth Annual Mathematics Tournament
April 15, 2023
Solutions

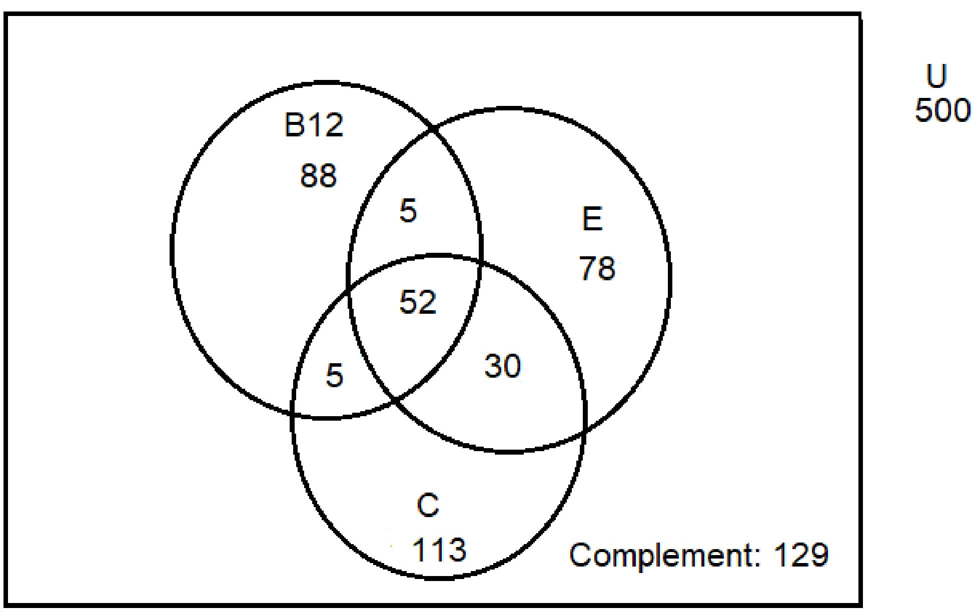
Round 1: Total number of tulip bulbs: 30, number of yellow tulip bulbs: 10, number of red tulip bulbs: 10. Let R : a randomly selected tulip bulb is red, Y : a randomly selected tulip bulb is yellow. So, we want to know $P(R_1 \cap Y_2) + P(Y_1 \cap R_2)$, where R_i, Y_i stands for red tulip bulb or yellow tulip bulb in the i^{th} draw, $i = 1, 2$.

$$P(R_1 \cap Y_2) + P(Y_1 \cap R_2) = P(Y_2 \mid R_1) \cdot P(R_1) + P(R_2 \mid Y_1) \cdot P(Y_1)$$
$$= \frac{10}{29} \cdot \frac{10}{30} + \frac{10}{29} \cdot \frac{10}{30} = \frac{10}{87} + \frac{10}{87} = \frac{20}{87}.$$

Round 2: Let $n(x)$ represent the number of elements in the set represented by x. Let U be the universal set.

$$n(B12) = 150, n(C) = 200, n(E) = 165$$
$$n(B12 \cap C) = 57, n(B12 \cap E) = 57, n(C \cap E) = 82$$
$$n(B12 \cap C \cap E) = 52$$
$$n(U) = 500$$
$$n(U) = n(B12) + n(C) + n(E) - n(B12 \cap C) - n(B12 \cap E) - n(C \cap E)$$
$$+ n(B12 \cap C \cap E) + n\left([B12 \cup C \cup E]^c\right)$$

So, $500 = 150 + 200 + 165 - 57 - 57 - 82 + 52 + n\left([B12 \cup C \cup E]^c\right) \Rightarrow n\left([B12 \cup C \cup E]^c\right) = 500 - 371 = 129$.



Round 3: F: original amount of grass grass in the field, G : amount of grass growing daily, C : amount of grass a cow eats daily

$$F + 14G = 14(60\text{C}) \Rightarrow F + 14G = 840\text{C}$$
$$F + 28G = 28(50\text{C}) \Rightarrow F + 28G = 1400C$$
$$14G = 560C \Rightarrow G = \frac{560}{14}C$$
$$\Rightarrow G = 40C$$

Hence, the maximum number of cows would be 40.

Round 4: Extend the sequence a little bit more and see the pattern.

$$3, 3, 2, 1, 3, 0, 3, 3, 2, 1, 3, 0 \ldots$$

So, we find a cycle of length 6. Dividing 1209 by 6, we will obtain the quotient of 201 with a remainder of 3. This represents 201 complete cycles of length 6 and 3 digits into the next cycle, which would give the digit 2 as the 1209^{th} term. Hence the 1209^{th} term is 2.

Round 5:

$$\text{Let } \log_9(p) = \log_{12}(q) = \log_{16}(p+q) = k.$$
$$\Rightarrow \log(p) = k \log 9 = 2k \log 3$$
$$\log(q) = k \log 12 = 2k \log 2 + k \log 3$$
$$\log(p+q) = k \log 16 = 4k \log 2.$$
$$\text{Hence, } \log(q) = \frac{1}{2}\log(p+q) + \frac{1}{2}\log p$$
$$\Rightarrow 2\log(q) = \log\left(p^2 + pq\right)$$
$$\Rightarrow q^2 = p^2 + pq \Rightarrow \left(\frac{q}{p}\right)^2 - \left(\frac{q}{p}\right) - 1 = 0$$
$$\Rightarrow \frac{q}{p} = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(1)(-1)}}{2(1)} = \frac{1 \pm \sqrt{5}}{2}$$
$$\text{So, } \frac{q}{p} = \frac{1+\sqrt{5}}{2} \text{ as } \frac{q}{p} > 0.$$

Round 6: For $n^2 - 2n - 8$ to be equal to a prime number, one of its factors must be equal to 1. Factoring gives $(n-4)(n+2)$, so either $n-4 = 1$ or $n+2 = 1$. Solving the first equation gives $n = 5$, and solving the second equation gives $n = -1$. $n = -1$ is not a natural number, thus not an answer. $n = 5$ is a natural number and makes $(n-4)(n+2) = (5-4)(5+2) = (1)(7) = 7$, which is a prime number. $n = 5$ is the only natural number.

Round 7: Let m, n be the numbers of digits in 2^{2005} and 5^{2005}. Then observe that

$$10^{m-1} < 2^{2005} < 10^{m} \quad , \quad 10^{n-1} < 5^{2005} < 10^{n}$$

implying that

$$10^{m+n-2} < 2^{2005} \cdot 5^{2005} = 10^{2005} < 10^{m+n}.$$

Hence $m+n-2 < 2005 < m+n$, so $m+n = 2006$, which is the answer.

Round 8: We wish to seat 5 people, 3 of whom must sit consecutively, in 12 seats. Since the block takes up 3 of the 12 places, it must begin in one of the first $12-(3-1) = 12-3+1 = 10$ positions. Once the block has been placed, there are $12-3=9$ seats left for the remaining $5-3=2$ people. They can be arranged in those seats in $P(9,2)$ ways. The people within the block can be arranged in 3! ways, which gives us the formula $10 \cdot P(9,2) \cdot 3! = 10 \cdot 72 \cdot 6 = 4320$ ways.

Note here $P(n,r)$ stands for the number of permutations of r items out of a pack of n items and is defined via $P(n,r) = \dfrac{n!}{(n-r)!}$, where $0! = 1, 1! = 1, 2! = 2 \cdot 1 = 2$, $3! = 3 \cdot 2 \cdot 1 = 6, 4! = 4 \cdot 3 \cdot 2 \cdot 1 = 24$, and in general $n! = n \cdot (n-1) \cdot 3 \cdot 2 \cdot \cdot 1$, n is a non negative integer.

Round 9: $\log_2 \left(\log_4 \left(\log_{\frac{1}{2}} \left(\log_9(2k) \right) \right) \right) = -1$

$$\begin{aligned}
&\Rightarrow \log_4 \left(\log_{\frac{1}{2}} \left(\log_9(2k) \right) \right) = \frac{1}{2} \\
&\Rightarrow \log_{\frac{1}{2}} \left(\log_9(2k) \right) = 2 \\
&\Rightarrow \quad \log_9(2k) = \frac{1}{4} \\
&\Rightarrow \quad 2k = 9^{\frac{1}{4}} = \sqrt{3} \\
&\Rightarrow \quad k = \frac{\sqrt{3}}{2}
\end{aligned}$$

Thus $\arccos(k) = \arccos\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{6}$.

Round 10: The diameter of the outside most circle is 6 units. So, radius is 3 units. Hence, the area of the outside most circle is 9π squared units. Each of the smaller inner circles has radius of 1 unit. So, the area of each inner circle is π square units. Thus we have $9\pi - 7\pi = 2\pi$. Hence, area of the shaded region is 2π square units.