Twenty-Sixth Annual Mathematics Tournament
April 15, 2023
Morning Component

Good morning!

- *Please do NOT open this booklet until given the signal to begin.*
- *There are 40 multiple choice questions. Answer the questions on the electronic grading form by giving the best answer to each question.*
- *The scoring will be done by giving one point for each question answered correctly and zero points for each question answered incorrectly or left blank. Thus, it is to your advantage to answer as many questions as possible, even if you have to guess. If there is a tie, question number 29 will be used again as a tie-breaker.*
- *This test was designed to be a CHALLENGE. It is difficult, and you may not have time to complete all questions. Do not worry if you are unable to answer several of the questions. Instead, we hope that you will obtain satisfaction from those questions which you ARE able to answer.*
- *You may write in the test booklet. You may keep your test booklet and any of your scrap papers. Only the electronic grading form will be collected and graded.*

Good luck!

Do Not Open Until Signaled

You may write in this test booklet. Only the Scantron sheet will be graded. Correct answers are awarded one point. Incorrect or blank answers are awarded 0 points.

1. Compute $\lim_{x \to -\infty} \dfrac{x^2(\sin x + \cos^3 x)}{(x^2+1)(x-3)}$.

   (a) 0

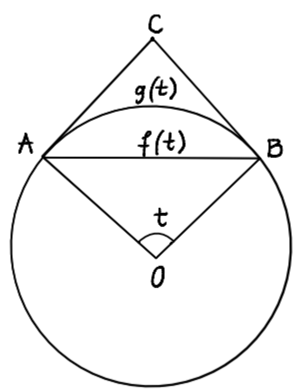
   (b) Does not exist

   (c) 2

   (d) -2

   (e) None of the above

2. Let t be the measure of a central angle $\angle AOB$ of a circle. The segments AC and BC are tangent to the circle at points A and B, respectively. The triangular region ΔABC is divided into the region outside the circle whose area is $g(t)$ and the region inside the circle with area $f(t)$. Compute $\lim_{t \to 0} \dfrac{f(t)}{g(t)}$.



   (a) 0

   (b) Does not exist

   (c) -1

   (d) 1

   (e) None of the above

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3. Compute the following limit, if it exists:

$$\lim_{x\to 0}\left(\frac{|x|}{x} - 5e^{x} + 6\right).$$

   (a) 1

   (b) Does not exist

   (c) ∞

   (d) 0

   (e) None of the above

4. Suppose a and b are real numbers such that $\lim_{x\to 0}\frac{\sin^2 x}{e^{ax} - bx - 1} = \frac{1}{2}$. Determine all possible ordered pairs (a, b).

   (a) $(2, 2)$

   (b) $(1, 1)$ and $(2, 2)$

   (c) $(-1, -1)$ and $(1, 1)$

   (d) $(2, 2)$ and $(-2, -2)$

   (e) None of the above

5. Compute $\lim_{x\to 0}(\ln|x| - \ln|\sin(x)|)$.

   (a) π

   (b) 0

   (c) 1

   (d) e

   (e) None of the above

6. Compute $\lim_{x \to 0} f'(x)$ given $f(x) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right) & x \neq 0 \\ 0 & x = 0 \end{cases}$.

   (a) 0

   (b) $\frac{1}{2}$

   (c) ∞

   (d) Does not exist

   (e) None of the above

7. Compute the following limit, if it exists: $\lim_{h \to 0} (1+h)^{\frac{x}{h}}$.

   (a) e^x

   (b) ∞

   (c) Does not exist

   (d) 0

   (e) None of the above

8. Compute the following limit, if it exists: $\lim_{x \to \infty} \frac{\sqrt{3x^2 + 3x} - \sqrt{3}x}{3}$.

   (a) $\frac{1}{3}$

   (b) $\frac{\sqrt{3}}{6}$

   (c) Does not exist

   (d) $3\sqrt{3}$

   (e) None of the above

9. Find the value of a that would make the function continuous for x in $\mathbb{R}$. $\mathbb{R}$ is the set of real numbers. $f(x) = \begin{cases} \dfrac{\sin(x)}{x\cos(x)} & x \neq 0 \\ a & x = 0 \end{cases}$.

   (a) 1

   (b) π

   (c) 0

   (d) There is no number

   (e) None of the above

10. If $f(x) = \begin{cases} (x^2)^{\left(\frac{x^2}{2}\right)} & x \neq 0 \\ 1 & x = 0 \end{cases}$, then what is $f'(0)$?

   (a) -2

   (b) -1

   (c) 0

   (d) 2

   (e) None of the above

11. Which of the following statements is correct?

   (a) If f is continuous at $x = c$, then f is differentiable at $x = c$.

   (b) If $\lim_{x \to c^-} f(x) = \infty$ and $\lim_{x \to c^+} f(x) = \infty$, then $\lim_{x \to c^-} f(x) = \lim_{x \to c^+} f(x)$.

   (c) If the function is differentiable at $x = c$, then it has a horizontal tangent line at $x = c$.

   (d) If $\lim_{x \to c^-} f(x)$ and $\lim_{x \to c^+} f(x)$ exist and $\lim_{x \to c^-} f(x) \neq \lim_{x \to c^+} f(x)$, then f has a non-removable discontinuity at $x = c$.

   (e) None of the above

12. Let $f(x) = \dfrac{x^{n+1}}{n+1}$, where n is a positive whole number. Find $f^{(n)}(x)$. $f^{(n)}(x)$ represents the n-th derivative of f with respect to x.

  (a) $n \cdot x$

  (b) $(n+1) \cdot x$

  (c) $n! \cdot x$

  (d) $(n+1)! \cdot x$

  (e) None of the above

13. Suppose that h is the composition of functions f with g; i.e., $h = f \circ g$ or $h(x) = f(g(x))$. If $f'(2) = 6, f(1) = 4, g(1) = 2,$ and $g'(1) = -2$, find $h'(1)$.

  (a) -12

  (b) -8

  (c) -24

  (d) -16

  (e) None of the above

14. Let $f(x) = \dfrac{4x+3}{x+2}$. Find $\left(f^{-1}\right)'(9)$.

  (a) 5

  (b) 0

  (c) -5

  (d) $\dfrac{1}{5}$

  (e) None of the above

15. Evaluate $f'(1)$ if $f(x) = \ln x^x$.

    (a) 0

    (b) 1

    (c) e

    (d) e^e

    (e) None of the above

16. Find the derivative of the function $f(x) = \sin\left(\sqrt{x}\,\right) + \sqrt{\cos(x)}$.

    (a) $\cos\left(\sqrt{x}\,\right) + \dfrac{1}{2\sqrt{\cos(x)}}$

    (b) $\dfrac{\cos\left(\sqrt{x}\,\right)}{2\sqrt{x}} - \dfrac{\sin(x)}{2\sqrt{\cos(x)}}$

    (c) $\cos\left(\sqrt{x}\,\right) - \dfrac{1}{2\sqrt{\cos(x)}}$

    (d) $\dfrac{\cos\left(\sqrt{x}\,\right)}{2\sqrt{x}} + \dfrac{\sin(x)}{2\sqrt{\cos(x)}}$

    (e) None of the above

17. Given $f(x) = \sin(\pi x) \cdot \cos(\pi x)$, for what values of x does f have a horizontal tangent line? In the answers $\mathbb{Z}$ represents the set of all integers.

    (a) $2k + 1, k \in \mathbb{Z}$

    (b) $k, k \in \mathbb{Z}$

    (c) $\dfrac{k}{2}, k \in \mathbb{Z}$

    (d) $\dfrac{2k+1}{4}, k \in \mathbb{Z}$

    (e) None of the above

18. Let N be the line normal to the graph of $y = x^2$ at the point $(-2, 4)$. At what other point does N meet the graph?

(a) $\left(\frac{3}{2}, \frac{9}{4}\right)$

(b) $\left(\frac{7}{4}, \frac{49}{16}\right)$

(c) $\left(\frac{9}{4}, \frac{81}{16}\right)$

(d) $\left(\frac{11}{4}, \frac{121}{16}\right)$

(e) None of the above

19. Consider the function $f(x) = a\sin(x) - \tan(x)$ defined on $[0, \pi/2)$. Find a such that the function has an inflection point at $\frac{\pi}{3}$.

(a) -16

(b) $-\frac{16}{3\sqrt{3}}$

(c) 16

(d) $-16\frac{\sqrt{3}}{2}$

(e) None of the above

20. If $f(1) = 10$ and $f'(x) \geq 2$ for $1 \leq x \leq 4$, what is the smallest possible value of $f(4)$?

(a) 8

(b) 10

(c) 14

(d) 16

(e) None of the above

21. Curve C is described by the equation $0.25x^2 + y^2 = 9$. Determine the $y-$coordinate(s) of the point(s) on the curve C whose tangent lines have slope equal to 1.

(a) $\frac{5\sqrt{3}}{3}$

(b) $-\frac{5\sqrt{3}}{3}$

(c) $\frac{3\sqrt{5}}{5}$ and $\frac{-3\sqrt{5}}{5}$

(d) $\frac{5\sqrt{3}}{3}$ and $\frac{-5\sqrt{3}}{3}$

(e) None of the above

22. A model for the cost, in dollars, of producing x units of an electronics item is:

$$C(x) = 1000 - 5x + \frac{1}{10}x^2, \quad 0 \leq x \leq 300.$$

Find the cost per unit when the average cost is minimized.

(a) \$15.00

(b) \$25.00

(c) \$37.50

(d) \$100.00

(e) None of the above

23. Let f be a continuous one-to-one function, such that $f(1) = 3$ and $f(7) = 8$. Assume $\int_1^7 f(x)dx = 31$. Calculate $\int_3^8 f^{-1}(x)dx$.

(a) 20

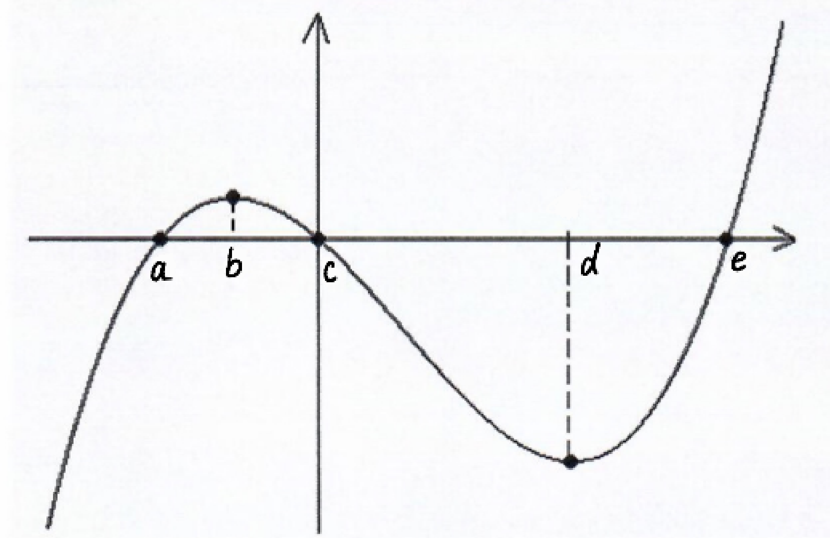
(b) 21

(c) 22

(d) Not enough information

(e) None of the above

24. The graph of a polynomial function f is shown below. If f' is the first derivative of f, then find the remainder when $f'(x)$ is divided by $x - b$.



(a) 0

(b) 1

(c) 2

(d) 3

(e) None of the above

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25. If $f(x) = \int_1^{x^3} \frac{1}{1+\ln t}\, dt$ for $x \geq 1$, then find $f'(2)$.

   (a) $\frac{1}{1+\ln 2}$

   (b) $\frac{12}{1+\ln 2}$

   (c) $\frac{1}{1+\ln 8}$

   (d) $\frac{12}{1+\ln 8}$

   (e) None of the above

26. Compute the integral $\int_0^4 e^{\sqrt{x}} dx$.

   (a) $e^2 + 1$

   (b) $2e^2 + 1$

   (c) $2e^2 + 2$

   (d) $e^2 + 2$

   (e) None of the above

27. Find the area between the curves $y = e^x$ and $y = e^{-x}$ from $x = -1$ to $x = 1$.

   (a) 0

   (b) $3e + \frac{2}{e} - 2$

   (c) $4e - \frac{5}{e}$

   (d) $2e + \frac{2}{e} - 4$

   (e) None of the above

28. Find the average value of $f(x) = \sin(x)\cos^4(x)$ over the interval $[0, \pi]$.

(a) $\frac{2}{5}$

(b) $\frac{1}{5}$

(c) $\frac{2}{5\pi}$

(d) $\frac{1}{5\pi}$

(e) None of the above

Reminder
Question 29 will be used as a tie-breaker, if necessary.

29. Find the value of a such that $\int_a^{\pi} \left(\frac{\cos(x)}{x} - \frac{\sin(x)}{x^2} \right) dx = -1$.

(a) 0

(b) $\frac{\pi}{2}$

(c) $\frac{\pi}{3}$

(d) $-\pi$

(e) None of the above

30. Compute $\lim_{x \to 0^+} (\tan x)^{x^2}$.

(a) ∞

(b) 0

(c) 1

(d) e

(e) None of the above

31. Which of the following integrals will yield the volume generated by rotating the region bounded by $y = x^2$ and $y = 2 - x^2$ about the line $x = 1$?

(a) $2\pi \int_{-1}^{1} (1 - x)(2 - 2x^2)dx$

(b) $2\pi \int_{0}^{1} (1 - x^2)^2 dx$

(c) $2\pi \int_{-1}^{1} x(2 - x^2)dx$

(d) $2\pi \int_{-1}^{1} \left[(2 - 2x^2)^2 - x^2\right] dx$

(e) None of the above

32. Determine the volume swept out by the region enclosed by the graph of $9x^2 + 4y^2 - 36x = 0$ when this region is rotated about the line $x = -2$.

(a) $24\pi^2$ cubic units

(b) $36\pi^2$ cubic units

(c) $48\pi^2$ cubic units

(d) $60\pi^2$ cubic units

(e) None of the above

33. Compute the integral $\int \frac{3x^2 + 2}{x^6 + 4x^4 + 2x^3 + 4x^2 + 4x + 1} dx$.

(a) $\frac{3}{x^4 + 6x - 7} + C$

(b) $-\frac{1}{x^3 + 2x + 1} + C$

(c) $\ln\left(x^6 + 4x^4 + 2x^3 + 4x^2 + 4x + 1\right) + C$

(d) $\frac{(3x^2 + 2)^2}{2} + C$

(e) None of the above

34. Compute the integral $\int_1^4 \frac{dx}{\sqrt{x}(3+\sqrt[4]{x})}$.

(a) $16 + 12\ln\left(\frac{4}{3+\sqrt{2}}\right)$

(b) $4\sqrt{2} - 4 + 12\ln\left(\frac{4}{3+\sqrt{2}}\right)$

(c) $4\sqrt{2} + 12\ln\left(\frac{3+\sqrt{2}}{4}\right)$

(d) $12\ln(3+\sqrt{2})$

(e) None of the above

35. $\int_2^4 \frac{x dx}{\sqrt{| 9 - x^2 |}} = \sqrt{5} + \sqrt{a}$, where $a =$ _____

(a) 0

(b) 5

(c) 7

(d) 12

(e) None of the above

36. A cable is hanging over the side of a 50-foot building. The end of the cable is 2 feet above the ground. The cable weighs 5 pounds per foot. How much work will be done to raise the cable to the roof?

(a) At most 5,000 pound feet.

(b) More than 5,000 pound feet but at most 10,000 pound feet.

(c) More than 10,000 pound feet but at most 20,000 pound feet.

(d) More than 20,000 pound feet.

(e) None of the above

37. Consider the area bounded by the graph of $y = \cos(x)$ and the x-axis over the domain $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. Determine the value $k \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ such that the line $x = k$ splits the total area into two parts where the area on the left is three times the area on the right.

(a) $\frac{\pi}{12}$

(b) $\frac{\pi}{6}$

(c) $\frac{\pi}{4}$

(d) $\frac{\pi}{3}$

(e) None of the above

38. You are standing on a road that intersects a railroad track at right angles, one-quarter of a mile from the intersection. You observe that the distance between you and an approaching train is decreasing at a constant rate of 25 miles per hour. How far from the intersection is the train when its speed is 40 miles per hour?

(a) 0 miles

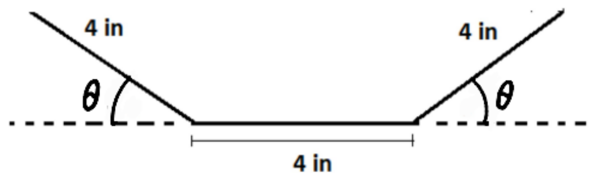
(b) $\frac{4}{5\sqrt{39}}$ miles

(c) $\frac{8}{5\sqrt{39}}$ miles

(d) $\frac{5}{4\sqrt{39}}$ miles

(e) None of the above

39. A rain gutter is to be constructed using a piece of aluminum, 12 inches wide. After marking a length of 4 inches from each edge, the piece of aluminum is bent up at an angle as shown. Find the angle θ, in radians, that will maximize the cross-sectional area of the rain gutter.



(a) $\frac{\pi}{6}$

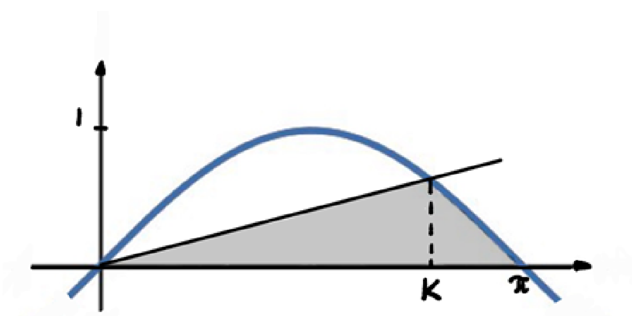
(b) $\frac{\pi}{5}$

(c) $\frac{\pi}{4}$

(d) $\frac{\pi}{3}$

(e) None of the above

40. In the figure below, half a period of $\sin(x)$ from 0 to π is split into two regions of equal area by a line through the origin. The line and the sine function intersect at a point whose x coordinate is K. What equation does K satisfy?



(a) $K\sin(K) + 2\cos(K) = 0$

(b) $\cos(K) = K\sin(K)$

(c) $K\cos(K) = \sin(K)$

(d) $(2 + K)\sin(K) + 2\cos(K) = 0$

(e) None of the above