University of North Georgia

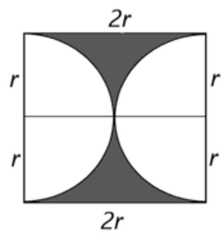
Mathematics Tournament

April 6, 2019

Solutions for the Afternoon Team Competition

Round 1

The area of the black region is found by taking the area of the square and subtracting the area of the semi-circles. The area of the black region is $(2r)^2 - 2\cdot\frac{\pi r^2}{2} = 4r^2 - \pi r^2 = 9 - 2.25\pi$. Solving we get that $r^2 = \frac{9}{4}$ and $r = \frac{3}{2}$. Therefore the perimeter of the square is $2r\cdot 4 = 2\cdot\frac{3}{2}\cdot 4 = 12$.



Round 2

The area of each grid is $(100\ ft)^2$.

$$\frac{1}{2}(400\ ft)h + (200\ ft)(100\ ft) = 0.5(13)(100\ ft)^2$$

$$(200\ ft)h + 2(100\ ft)^2 = 6.5(100\ ft)^2$$

$$(200\ ft)h = 4.5(100\ ft)^2 = \frac{9}{2}(100\ ft)^2$$

$$h = \frac{9}{4}(100\ ft) = 225\ ft$$

$$L = \sqrt{(400\ ft)^2 + (225\ ft)^2} = \sqrt{210625\ ft^2} = 458.9\ ft$$

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Round 3

The median is the average of the two middle numbers. The 1010th and 1011th numbers of the set of even integer numbers are middle numbers and their average is 2020. If x is the 1010th number, then $x+2$ is the 1011th number. We have $\frac{x+(x+2)}{2}=2021$, thus the 1010th number is 2020 and the 1011th number 2022. Beside the 1010th and 1011th numbers, there are 2018 other numbers in the list. There are 1010 numbers on each side of the median. The smallest term in the list can be found by subtracting 2 from the 1010th number 1099 times. Thus the smallest even number in the list is given by $2020-1009\cdot 2=2$. Similarly, the greatest number is found by adding 2 to the 1011th number 1009 times. Thus the largest even number in the list is given by $2022+1009\cdot 2=4040$. Let $S=2+4+6+\cdots+4038+4040$. Then $2S=(2+4040)+(4+4038)+\cdots+(4038+4)+(4040+2)$. This gives $2S=4042\cdot 2020=8{,}164{,}840$ and $S=4{,}082{,}420$.

Round 4

Using laws of logarithms we have $\log_{10}\left(x^2-16\right)\leq 2$, and then $x^2-16\leq 100$ and $x^2\leq 116$. The integer solutions would be $x=-10,-9,-8,\ldots,8,9,10$, but $\log_{10}(x+4)+\log_{10}(x-4)$ is only defined for $x>4$, so the sum of the integer solutions is $5+6+7+8+9+10=45$.

Round 5

5 consecutive years has $5(365)+2=1827$ maximum days. There are 0, 1, 2, or 3 possibilities for ingredients. Let n be the number of topping ingredients that must be available. The number of different pizzas will either have 0, 1, 2, or 3 ingredients. That is

(# of pizzas with 0 ingredients) + (# of pizzas with 1 ingredient) + (# of pizzas with 2 ingredients) + (# of pizzas with 3 ingredients) = $\binom{n}{0}+\binom{n}{1}+\binom{n}{2}+\binom{n}{3}\geq 1827$. This gives $1+n+\frac{n(n-1)}{2}+\frac{n(n-1)(n-2)}{6}\geq 1827$. Simplifying gives $f(n)=n^3+5n\geq 10956$ and $f(22)<10956<f(23)$, hence the least n is 23.

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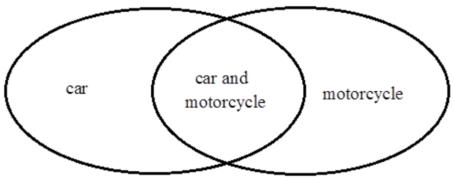
Round 6

The number of families that have only motorcycle is $200-150=50$. In the 150 families that own car, half of them own both car and motorcycle, so the number of families that have both car and motorcycle is $\frac{1}{2}(150)=75$. The number of families that have motorcycle are $50+75=125$.



Round 7

Note $\sqrt{7+4\sqrt{3}}=\sqrt{\left(2+\sqrt{3}\right)^2}=2+\sqrt{3}$ and $\sqrt{7-4\sqrt{3}}=\sqrt{\left(2-\sqrt{3}\right)^2}=2-\sqrt{3}$. The equation can be rewritten as $(2+\sqrt{3})^{\cos x}+(2-\sqrt{3})^{\cos x}=4$.

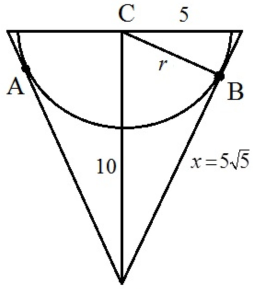
Moreover, since $2-\sqrt{3}=\frac{1}{2+\sqrt{3}}$, then we have $(2+\sqrt{3})^{\cos x}+(2+\sqrt{3})^{-\cos x}=4$ or $(2+\sqrt{3})^{2\cos x}-4(2+\sqrt{3})^{\cos x}+1=0$.

The quadratic equation above has solution $(2+\sqrt{3})^{\cos x}=2+\sqrt{3}$, $(2+\sqrt{3})^{\cos x}=2-\sqrt{3}$ or $\cos x=\pm 1$. It implies that $x=k\pi$, where k is any integer.

Round 8

Let the radius of the ice cream scoop be $r=5$. Since the cone is symmetric, the points of tangency *A* and *B* of the ice cream scoop to the cone make a circle on the inside of the cone. The center of the ice cream scoop must coincide with the center of the base of the cone (Figure). From the figure, angle *CBD* is a right angle, so we have $x^2=5^2+10^2$ => $x=5\sqrt{5}$. Since *CAD* and *CBD* are similar triangles,

$\frac{5}{5\sqrt{5}}=\frac{r}{10}$ => $10(5)=5\sqrt{5}r$ => $r=\frac{10}{\sqrt{5}}\cdot\frac{\sqrt{5}}{\sqrt{5}}$ => $r=2\sqrt{5}$



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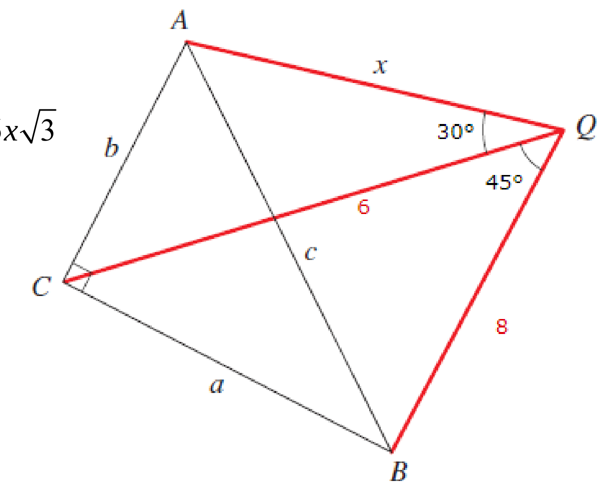
Round 9

Using the law of cosines we have: $a^2 = 36+64-48\sqrt{2}$, $b^2 = x^2+36-6x\sqrt{3}$, and $c^2 = x^2+64-16x\cos\left(30^\circ+45^\circ\right)$, where $\cos\left(30^\circ+45^\circ\right)=\left(\sqrt{6}-\sqrt{2}\right)/4$. Using the Pythagorean Theorem we have:



$$c^2 = a^2+b^2$$

$$\Rightarrow x^2+64-\frac{16x\left(\sqrt{6}-\sqrt{2}\right)}{4} = 36+64-48\sqrt{2}+x^2+36-6x\sqrt{3}$$

$$\Rightarrow \frac{-16x\left(\sqrt{6}-\sqrt{2}\right)}{4} = 72-48\sqrt{2}-6x\sqrt{3}$$

$$\Rightarrow -4x\left(\sqrt{6}-\sqrt{2}\right)+6x\sqrt{3} = 72-48\sqrt{2}$$

$$\Rightarrow x\left(-4\left(\sqrt{6}-\sqrt{2}\right)+6\sqrt{3}\right) = 72-48\sqrt{2}$$

$$\Rightarrow x = \frac{72-48\sqrt{2}}{-4\sqrt{6}+4\sqrt{2}+6\sqrt{3}}$$

$$\Rightarrow \quad x = 0.66$$

Round 10

Let $\frac{1}{x}+\frac{1}{x^2}+\frac{1}{x^3}+\ldots = s$. Then factoring out $\frac{1}{x}$ gives us $\frac{1}{x}\left(1+\frac{1}{x}+\frac{1}{x^2}+\frac{1}{x^3}+\ldots\right)=s$.

Since $1+\frac{1}{x}+\frac{1}{x^2}+\frac{1}{x^3}+\ldots = 2019$, we have $1+\frac{1}{x}\square 2019 = 2019$ => $\frac{1}{x}=\frac{2018}{2019}$.

Adding 1 to both sides of the second equation gives us $1+\frac{1}{y}+\frac{1}{y^2}+\frac{1}{y^3}+\ldots = 2018$.

Then $1+\frac{1}{y}\square 2018 = 2018$ => $\frac{1}{y}=\frac{2017}{2018}$.

Then $\left(\frac{1}{x}\right)\div\left(\frac{1}{y}\right)=\frac{(2018)^2}{(2019)(2017)}$.

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