UNG | UNIVERSITY of NORTH GEORGIA

Twenty-Seventh Annual Mathematics Tournament
April 6, 2024
Morning Component

Good morning!

- *Please do NOT open this booklet until given the signal to begin.*
- *There are 40 multiple choice questions. Answer the questions on the electronic grading form by giving the best answer to each question.*
- *The scoring will be done by giving one point for each question answered correctly and zero points for each question answered incorrectly or left blank. Thus, it is to your advantage to answer as many questions as possible, even if you have to guess. If there is a tie, question number 23 will be used as a tie-breaker. If the tie remains, question number 37 will be used as the second tie-breaker.*
- *This test was designed to be a CHALLENGE. It is difficult, and you may not have time to complete all questions. Do not worry if you are unable to answer several of the questions. Instead, we hope that you will obtain satisfaction from those questions which you ARE able to answer.*
- *You may write in the test booklet. You may keep your test booklet and any of your scrap papers. Only the electronic grading form will be collected and graded.*

Good luck!

Do Not Open Until Signaled

You may write in this test booklet. Only the Scantron sheet will be graded. Correct answers are awarded one point. Incorrect or blank answers are awarded 0 points.

1. Recall that $e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$. Compute $\lim_{n\to\infty}\left(1+\frac{1}{2n}\right)^n$.

   (a) $2e$

   (b) $\frac{e}{2}$

   (c) $\sqrt{e}$

   (d) e^2

   (e) None of the above

2. Let $f(x) = x^x$. What is $f'(1)$?

   (a) 1

   (b) $\ln(1)$

   (c) -1

   (d) e

   (e) None of the above

3. Given that $\lim_{x\to 1}\frac{f(x)-4}{2x-2} = 7$, what is $\lim_{x\to 1} f(x)$?

   (a) 0

   (b) 4

   (c) 7

   (d) 11

   (e) None of the above

4. Let $f(x) = \tan(\sqrt{x})$. What is $f'(x)$?

   (a) $\dfrac{1}{2\sqrt{x}\cos^2(\sqrt{x})}$

   (b) $\dfrac{\cos^2(x)}{2\sqrt{x}}$

   (c) $\cos^2(\sqrt{x}) - \dfrac{1}{2\sqrt{\cos(x)}}$

   (d) $\dfrac{\cos(\sqrt{x})}{2\sqrt{x}}$

   (e) None of the above

5. Compute $\lim_{\Delta x \to 0} \dfrac{(\Delta x + 2)\cos(\Delta x) - 2}{\Delta x}$.

   (a) $\dfrac{\sqrt{3}}{2}$

   (b) 0

   (c) 1

   (d) 2

   (e) None of the above

6. Find a formula for the n^{th} derivative of $y = (a + bx)^n$.

   (a) $f^{(n)}(x) = 0$

   (b) $f^{(n)}(x) = bn(a + bx)^{n-1}$

   (c) $f^{(n)}(x) = n!b^n$

   (d) $f^{(n)}(x) = n^n b^n$

   (e) None of the above

7. Compute the derivative.

$$\frac{\mathrm{d}}{\mathrm{d}\,x}\left(\int_0^{2x} \sin(2t)\cos(2t)\ \mathrm{d}t\right).$$

(a) $4\sin(4x)\cos(4x)$

(b) $2\sin(4x)\cos(4x)$

(c) $2\sin(2x)\cos(2x)$

(d) $2\sin^2(t)\cos(t)$

(e) None of the above

8. Let $f(x) = \tan(x), -\frac{\pi}{2} < x < \frac{\pi}{2}$, and g be the inverse of f. Compute the value of $g'(0)$.

(a) 1

(b) 17

(c) 0

(d) $\sqrt{17}$

(e) None of the above

9. Compute $\lim_{x\to 0}\frac{\cos x \tan x}{x}$.

(a) 1

(b) -1

(c) 0

(d) Does not exist

(e) None of the above

10. Compute $\lim_{x\to a}\left[\left(\frac{1}{x}-\frac{1}{a}\right)\left(\frac{1}{x-a}\right)\right]$.

(a) $\frac{1}{a}$

(b) Does not exist

(c) a^{-2}

(d) $-\frac{1}{a^2}$

(e) None of the above

11. Compute $\lim_{x\to\infty}\left(xe^{\frac{1}{x}}-x\right)$.

(a) Does not exist

(b) 1

(c) 0

(d) -1

(e) None of the above

12. Let $f(x)=\begin{cases} Ax-B & x\leq 1 \\ kx & 1<x<2 \\ Bx^2-A & x\geq 2 \end{cases}$. Determine conditions on A and B such that $f(x)$ is continuous at $x=1$ but not at $x=2$.

(a) $A-B=k$ and $B\neq k$

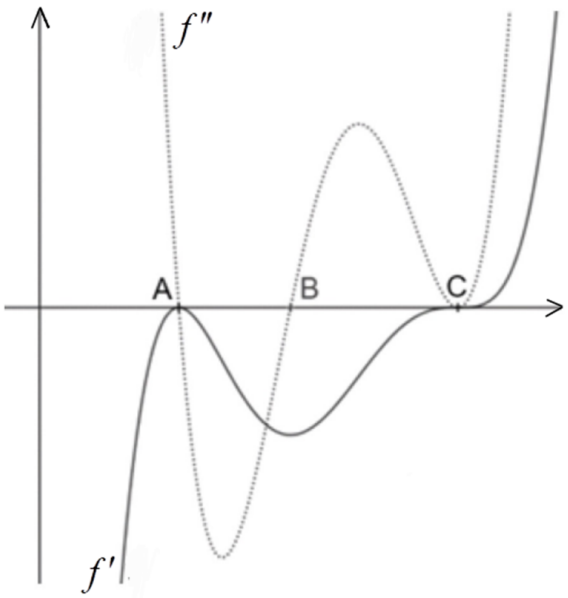
(b) $A+B\neq k$ and $B=2k$

(c) All values of $A, B,$ and k make $f(x)$ continuous at $x=1$ and discontinuous at $x=2$.

(d) There are no possible conditions for $A, B,$ and k that make $f(x)$ continuous at $x=1$ and discontinuous at $x=2$.

(e) None of the above

13. Given the graphs of f' and f'' below, where f' is represented by the solid curve and f'' is represented by the dotted curve, at which point(s) does the graph of f have a relative extrema?



(a) f only has a relative minimum and it occurs at A

(b) f only has a relative maximum and it occurs at A

(c) f has relative maximum at A and relative minimum at C

(d) f only has a relative minimum and it occurs at C

(e) None of the above

14. Given the graphs of f' and f'' in problem number 13, how many inflection point(s) does the graph of f have?

(a) 0

(b) 1

(c) 2

(d) 3

(e) None of the above

15. Compute $f'(0)$, where $f(x) = x\tan(x)$.

(a) 1

(b) -1

(c) 0

(d) Does not exist

(e) None of the above

16. Find a number a that would make $f(x) = \begin{cases} \dfrac{3x^2\cos(x)}{\sin(x)} & \text{for } x \neq 0 \\ a & \text{for } x = 0 \end{cases}$ continuous on the interval $\left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right]$.

(a) 3

(b) 1

(c) There is no such a.

(d) 0

(e) None of the above

17. Let f be a one-to-one continuous function, such that $f(3) = 6$ and $f(5) = 4$. Assume $\int_3^5 f(x)\ \mathrm{d}x = 9$. Compute $\int_4^6 f^{-1}(x)\ \mathrm{d}x$.

(a) 5

(b) 7

(c) 9

(d) 11

(e) None of the above

18. Let $f(x) = \sqrt{x + \sqrt{x}}$, find $f'(1)$.

(a) $\frac{3\sqrt{2}}{8}$

(b) $\frac{\sqrt{2}}{8}$

(c) $\frac{1}{2}$

(d) 1

(e) None of the above

19. Let $f(x) = x^n - x^{n-2}$ where f is an n^{th}-degree polynomial with $n > 2$. What is the maximum number of turning points?

(a) 2

(b) 3

(c) $n - 1$

(d) Can not be determined

(e) None of the above

20. If $f(x) = x \sin(x)$, then compute $f^{(100)}(0)$.

(a) 100

(b) -100

(c) 102

(d) -98

(e) None of the above

21. Compute the value of $\int_3^7 (1 - 5f(x))\ \mathrm{d}x$, provided $\int_3^8 f(x)\ \mathrm{d}x = 10$ and $\int_7^8 f(x)\ \mathrm{d}x = 8$.

(a) 0

(b) -6

(c) 4

(d) -4

(e) None of the above

22. $f(x) = \dfrac{\ln(x^2)}{x}$ has two critical numbers. One of them is $x = e$. What is the other one?

(a) $x = -e$

(b) $x = e^2$

(c) $x = 2e$

(d) $x = -\dfrac{e}{2}$

(e) None of the above

Reminder: Question 23 will be used as the first tie-breaker, if necessary.

23. Compute the value of $\int_0^{\pi/2} |\sin(x) - \cos(x)|\ \mathrm{d}x$.

(a) $2\left(\sqrt{2} - 1\right)$

(b) $\sqrt{2} - 1$

(c) $2\left(\sqrt{2} + 1\right)$

(d) $\sqrt{2} + 1$

(e) None of the above

24. A light is at the top of a pole 50 feet high. A ball is dropped from the same height from a point 30 feet away from the light. How fast is the shadow of the ball moving along the ground $\frac{1}{2}$ second later? Assume the ball falls a distance of $x = 16t^2$ feet in t seconds.

    (a) 1500 feet per sec

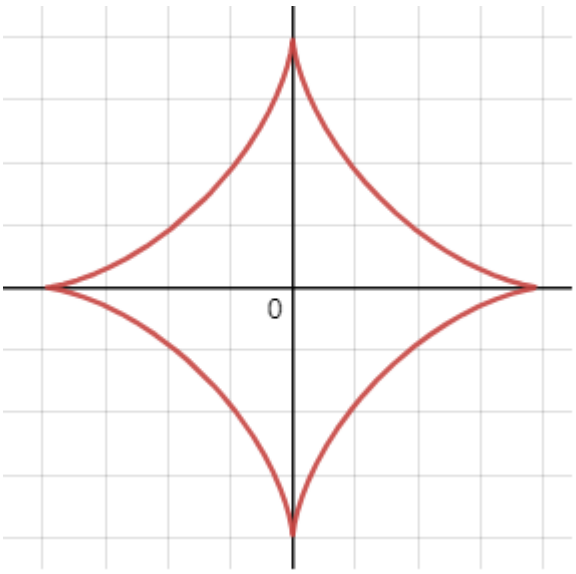
    (b) 3000 feet per sec

    (c) 1000 feet per sec

    (d) 2250 feet per sec

    (e) None of the above

25. Calculate the length of the astroid $x^{\frac{2}{3}} + y^{\frac{2}{3}} = 4$ as shown in the figure below. (The length of an astroid is the sum of the four arc lengths.)



    (a) 6 units

    (b) 24 units

    (c) 48 units

    (d) 12 units

    (e) None of the above

26. Compute the volume of the solid formed by rotating the region under the curve $y = \dfrac{1}{x^2+1}$, bounded by the x-axis, and the lines $x = 0$ to $x = 3$ about the y-axis.

(a) $\pi \ln(2)$ cubic units

(b) $\pi \ln(5)$ cubic units

(c) $\pi \ln(10)$ cubic units

(d) $\pi \ln(4)$ cubic units

(e) None of the above

27. Compute the volume of the solid formed by rotating the triangular region determined by the points $(0, 1)$, $(1, 1)$ and $(1, 3)$ about the line $x = 3$.

(a) 14π cubic units

(b) 2π cubic units

(c) $\dfrac{14}{3}\pi$ cubic units

(d) $\dfrac{14}{3}$ cubic units

(e) None of the above

28. What is the average value of f where $f(x) = \ln(x)$ over the interval $[1, e]$?

(a) $\dfrac{1}{2}$

(b) $\dfrac{1}{2}(e+1)$

(c) $\dfrac{1}{2}(e-1)$

(d) $\dfrac{1}{e-1}$

(e) None of the above

29. Compute the value of $\int_{-3}^{2} x(x+3)^{1/2}\,\mathrm{d}x$.

(a) 0

(b) $\frac{2\sqrt{5}}{5}$

(c) $-\frac{2\sqrt{5}}{5}$

(d) $\frac{2}{5}$

(e) None of the above

30. Compute $\lim_{x\to 0} \frac{1}{x^3}\int_0^x \frac{t^2}{t^4+1}\,\mathrm{d}t$.

(a) 1

(b) $\frac{1}{2}$

(c) $\frac{1}{3}$

(d) $\frac{1}{4}$

(e) None of the above

31. Find the sum of the x-coordinate and y-coordinate of the x and y intercepts, respectively, of any line tangent to the graph of $x^{\frac{1}{2}} + y^{\frac{1}{2}} = a^{\frac{1}{2}}$.

(a) $a^{\frac{1}{2}}$

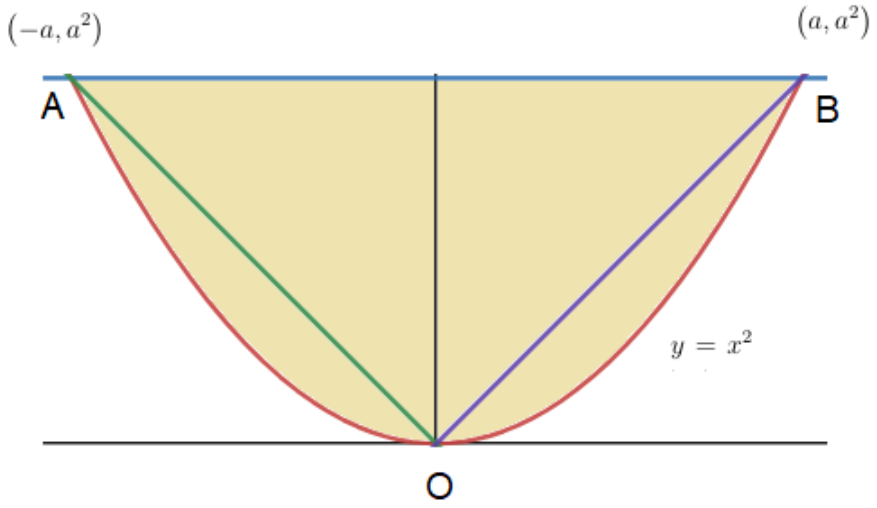
(b) Cannot be determined

(c) 1

(d) a

(e) None of the above

32. The figure below shows a triangle ΔAOB and region bounded below the line $y = a^2$ and above the parabola $y = x^2$. Find the limit, as a approaches 0, of the ratio of the area of the triangle ΔAOB to the area of the shaded region bounded by the horizontal line and the parabola.



(a) $\frac{1}{2}$

(b) $\frac{3}{4}$

(c) $\frac{7}{8}$

(d) 1

(e) None of the above

33. If $f(x) = \dfrac{e^x - 1}{e^x + 1}$, then

(a) f is even and the range of f is $(-1, 1)$.

(b) f is even and the range of f is $(-\infty, \infty)$.

(c) f is odd and the range of f is $(-1, 1)$.

(d) f is neither odd nor even and the range of f is $(-1, 1)$.

(e) None of the above

34. Find the slope of the tangent line to the curve implicitly defined by the equation $y^4 - xy^2 + x^4 = 1$ at the point $(1, 1)$.

(a) 0

(b) $-\frac{1}{3}$

(c) $-\frac{3}{2}$

(d) -1

(e) None of the above

35. Compute the area of the region bounded by the straight lines $x = \frac{1}{2}$ and $x = 2$, and the curves given by the equations $y = \ln(x)$ and $y = 2^x$.

(a) $\frac{1}{\ln 2}(4 - \sqrt{2}) - \frac{5}{2}\ln 2 + \frac{3}{2}$

(b) $\frac{1}{\ln 2}(4 - \sqrt{2}) - \frac{5}{2}\ln 2$

(c) $\frac{1}{\ln 2}(4 + \sqrt{2}) - \frac{5}{2}\ln 2 + \frac{3}{2}$

(d) $\frac{1}{\ln 2}(4 + \sqrt{2}) - \frac{5}{2}\ln 2$

(e) None of the above

36. Compute $\frac{\mathrm{d}y}{\mathrm{d}x}$ for $y = (\sin(x))^{\tan(x)}$.

(a) $\tan(x)(\sin(x))^{\tan(x)-1}\cos(x)$

(b) $(\sin(x))^{\tan(x)}\left[1 + \sec^2(x)\ln(\sin(x))\right]$

(c) $(\sin(x))^{\tan(x)}\sec^2(x)\ln(\sin(x))$

(d) $\tan(x)(\sin(x))^{\tan(x)-1}$

(e) None of the above

Reminder: Question 37 will be used as the second tie-breaker, if necessary.

37. Let f be a differentiable function defined by $f(x) = \dfrac{x^2 - ax + 1}{x^2 + ax + 1}$, where $0 < a < 2$. Let $g(x) = \displaystyle\int_0^{e^x} \frac{f(t)}{1+t^2}\,\mathrm{d}t$. Which of the following is true?

(a) $g'(x)$ is positive on $(-\infty, 0)$ and negative on $(0, \infty)$

(b) $g'(x)$ is negative on $(-\infty, 0)$ and positive on $(0, \infty)$

(c) $g'(x)$ changes sign on both $(-\infty, 0)$ and $(0, \infty)$

(d) $g'(x)$ is positive on $(-\infty, \infty)$

(e) None of the above

38. Compute the value of $\displaystyle\int \frac{3x^2 - 5x + 1}{x - 2}\,\mathrm{d}x$.

(a) $\dfrac{x^3 - \dfrac{5x^2}{2} + x}{\dfrac{x^2}{2} - 2x} + C$

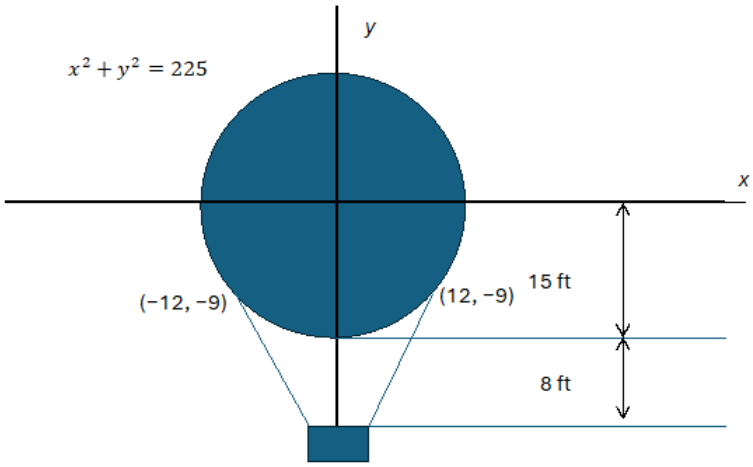
(b) $\dfrac{x^3 - 5x^2 + x}{x^2 - 2x} + C$

(c) $\dfrac{3x^2}{2} - x - 3\ln|x - 2| + C$

(d) $\dfrac{3x^2}{2} + x + 3\ln|x - 2| + C$

(e) None of the above

39. The designer of a 30-feet diameter spherical hot-air balloon wishes to suspend the gondola 8 feet below the bottom of the balloon with suspension cables tangent to the surface of the balloon. Two of the cables are shown running from the top edges of the gondola to their points of tangency $(-12, -9)$ and $(12, -9)$. How wide must the gondola be?



(a) 2.5 feet

(b) 3 feet

(c) 3.5 feet

(d) 4 feet

(e) None of the above

40. Let $f(x) = \begin{cases} 1 & x < 1 \\ -x^2 + 4x - 2 & 1 \leq x < 3 \\ 2 & x = 3 \\ \frac{3}{5}x - \frac{4}{5} & x > 3 \end{cases}$. Compute $\lim_{x \to 3} f(f(x))$.

(a) -1

(b) 1

(c) 0

(d) Does not exist

(e) None of the above