UNG | UNIVERSITY of NORTH GEORGIA™

# Twenty-Eighth Annual Mathematics Tournament
## April 5th, 2025

# Morning Component

# Good morning!

- *Please do NOT open this booklet until given the signal to begin.*
- *There are 40 multiple choice questions. Answer the questions on the electronic grading form by giving the best answer to each question.*
- *The scoring will be done by giving one point for each question answered correctly and zero points for each question answered incorrectly or left blank. Thus, it is to your advantage to answer as many questions as possible, even if you have to guess. If there is a tie, question number 3 will be used as a tie-breaker. If the tie remains, question number 28 will be used as the second tie-breaker. In case of a further tie, question number 38 will be used as the third tie-breaker.*
- *This test was designed to be a CHALLENGE. It is difficult, and you may not have time to complete all questions. Do not worry if you are unable to answer several of the questions. Instead, we hope that you will obtain satisfaction from those questions which you ARE able to answer.*
- *You may write in the test booklet. You may keep your test booklet and any of your scrap papers. Only the electronic grading form will be collected and graded.*

# Good luck!

# Do Not Open Until Signaled

**You may write in this test booklet. Only the Scantron sheet will be graded. Correct answers are awarded one point. Incorrect or blank answers are awarded 0 points.**

1. Compute $\lim_{x\to\infty} e^{\sin(2x)-x}$.

   (a) 1

   (b) $\infty$

   (c) 0

   (d) Does not exist

   (e) None of the above

2. Find $A$ so that the following integral $\int_{-2A}^{A} \left(\frac{2}{3}x+2\right) \mathrm{d}x$ is equal to 9.

   (a) $-1$

   (b) 1

   (c) $-2$

   (d) 2

   (e) None of the above

3. **First Tie-Breaker:** Let $f(x) = \int_0^{\sqrt{x}} t^2 \, \mathrm{d}t$, for $x \geq 0$. Find $f'(4)$.

   (a) 0

   (b) 1

   (c) 2

   (d) 4

   (e) None of the above

4. If $f(t) = (\ln t)^2, t > 0$, then:

   (a) $\lim_{t\to\infty} \frac{f(t)}{t}$ does not exist

   (b) $\lim_{t\to\infty} \frac{f'(t)}{t} = 2$

   (c) $\lim_{t\to\infty} (f(t+1) - f(t)) = 0$

   (d) $\lim_{t\to\infty} (f(t+1) - f(t))$ does not exist

   (e) None of the above

5. If $f$ is a continuous function, then $\int_0^1 f(1-x)\,\mathrm{d}x$ is equal to:

   (a) $-\int_0^1 f(x)\mathrm{d}x$

   (b) $\int_0^1 f(x)\mathrm{d}x$

   (c) $\int_0^1 f(-x)\mathrm{d}x$

   (d) $-\int_0^1 f(-x)\mathrm{d}x$

   (e) None of the above

6. Find the $n$th derivative for $f(x) = x^n e^x$ at $x = 0$.

   (a) $n(n+1)$

   (b) $(n+1)!$

   (c) 0

   (d) $n!$

   (e) None of the above

7. Determine the area of the region in the first quadrant, bounded by the graphs of the functions

$$y = \sin x, y = \cos(2x), \text{ and the } y\text{-axis.}$$

(a) $\dfrac{3\sqrt{3}}{4}$

(b) $\dfrac{3\sqrt{3}}{2}$

(c) $\dfrac{3\sqrt{3}}{2} - 1$

(d) $\dfrac{3\sqrt{3}}{4} - 1$

(e) None of the above

8. Compute the integral $\displaystyle\int_0^{\ln 2} \sqrt{e^x - 1}\, dx$.

(a) $1 - \dfrac{\pi}{4}$

(b) $2 - \dfrac{\pi}{2}$

(c) $2 + \dfrac{\pi}{2}$

(d) $2 - \dfrac{\pi}{4}$

(e) None of the above

9. Where does the equation $xy = e^x$ have a vertical tangent line?

(a) $x = 0$

(b) $x = 1$

(c) $x = 2$

(d) $x = 2e$

(e) None of the above

10. Compute the integral $\int_0^\infty \frac{4+x}{e^x}\,\mathrm{d}x$.

(a) $e+1$

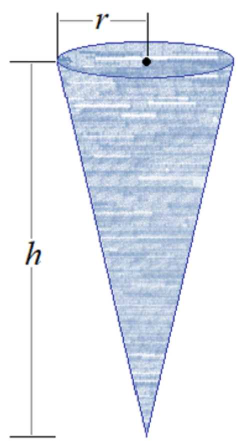
(b) $\frac{4}{e}$

(c) 5

(d) $\infty$

(e) None of the above

11. A cone-shaped icicle is dripping from a roof, as illustrated. The radius of the icicle is decreasing at a rate of 0.25 centimeters per hour, while the height is increasing at a rate of 0.75 centimeters per hour. At the point in time where the radius of the icicle is 6 centimeters and the height is 22 centimeters, what is the rate of decrease in the volume of the icicle?




(a) $13\pi$ cubic centimeters per hour

(b) $20.5\pi$ cubic centimeters per hour

(c) $23.5\pi$ cubic centimeters per hour

(d) $31\pi$ cubic centimeters per hour

(e) None of the above

12. Compute the integral $\int \frac{1}{1+\sin(x)}\,\mathrm{d}x$.

    (a) $\tan(x) + \sec(x)\sin(x) + C$

    (b) $\cot(x) + \sin(x) + C$

    (c) $\tan(x) - \sin(x) + C$

    (d) $\tan(x) - \sec(x) + C$

    (e) None of the above

13. For what value of the constant $m$, if any, is $f(x) = \begin{cases} \sin(4x), & x \leq 0 \\ mx, & x > 0 \end{cases}$ differentiable at $x = 0$?

    (a) 1

    (b) 2

    (c) 4

    (d) 8

    (e) None of the above

14. Compute $\lim_{x \to 0} \frac{\tan x - \sin x}{x^3}$, if exists.

    (a) $\frac{1}{6}$

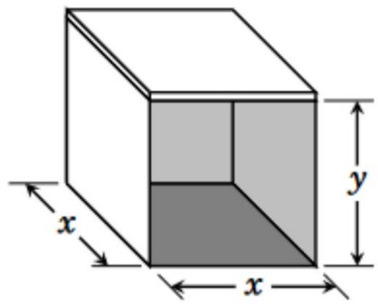
    (b) $\frac{1}{3}$

    (c) $\frac{1}{2}$

    (d) Does not exist

    (e) None of the above

15. You need to build a shed with an open front and square base (as illustrated), and containing a volume of $10,000$ cubic feet.




The costs of materials are:

Roof: \$10 per square foot;
Floor: \$5 per square foot;
Walls: \$8 per square foot;

Find the dimensions $x$ and $y$ that will minimize the total costs of materials.

(a) $x = 20$ ft. and $y = 25$ ft.

(b) $x = 20$ ft. and $y = 30$ ft.

(c) $x = 30$ ft. and $y = 30$ ft.

(d) $x = 25$ ft. and $y = 25$ ft.

(e) None of the above

16. If $f(x) = 1 + \dfrac{1}{x}\displaystyle\int_1^x f(t)\,\mathrm{d}t$, then compute the value of $f(e)$.

(a) 2

(b) 0

(c) $-2$

(d) 1

(e) None of the above

17. Determine the area of the region bounded by the graph of $y = \frac{\ln x}{x^2}$, the $x$-axis, and the line $x = e$.

    (a) $\frac{e}{2}$

    (b) $\frac{2}{e}$

    (c) $\infty$

    (d) 1

    (e) None of the above

18. Compute $\lim_{x \to 0} \frac{|2x-1| - |2x+1|}{x}$.

    (a) 0

    (b) Does not exist

    (c) 4

    (d) $-4$

    (e) None of the above

19. Find the $n$th derivative of the function $g(x) = \frac{1}{x+2}$.

    (a) $g^{(n)}(x) = \frac{(-1)^{n+1} n!}{(x+2)^{n+1}}$

    (b) $g^{(n)}(x) = \frac{(-1)^n n!}{(x+2)^n}$

    (c) $g^{(n)}(x) = \frac{(-1)^n (n+1)!}{(x+2)^n}$

    (d) $g^{(n)}(x) = \frac{(-1)^n n!}{(x+2)^{n+1}}$

    (e) None of the above

20. Find $a$ such that $\frac{\mathrm{d}}{\mathrm{d}a}\left(\int_0^{\sqrt{a}} \sin(t^2)\ \mathrm{d}t\right) = 0$.

(a) $a = 0$

(b) $a = k\pi$ where $k$ is an integer

(c) $a = k\pi$ where $k$ is a natural number

(d) $a = k\pi$ where $k$ is a non-zero integer

(e) None of the above

21. Suppose that $\int_0^x f(t)\ \mathrm{d}t = \sin x$. Find $f(\pi)$.

(a) 1

(b) $-1$

(c) 0

(d) $\frac{\sqrt{3}}{2}$

(e) None of the above

22. Compute the integral $\int_0^{\pi/2} \frac{\mathrm{d}x}{1+\tan^{2025}(x)}$.

(a) $\frac{\pi}{2}$

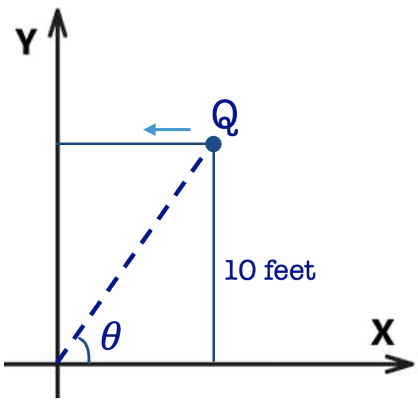
(b) $\pi$

(c) $\frac{\pi}{4}$

(d) 0

(e) None of the above

23. An object Q is moving in the first quadrant of the plane. Its motion is parallel to the $x$-axis; its distance to the $x$-axis is always 10 feet. Its velocity is 3 feet per second to the left. We define $\theta$ to be the angle between the positive $x$-axis and the line segment from the origin to the object Q. Compute the rate of change of the angle $\theta$ at the moment when $\theta = \frac{\pi}{3}$.




(a) $\dfrac{1}{2}$ rad/s

(b) $\dfrac{3}{40}$ rad/s

(c) $\dfrac{1}{\pi}$ rad/s

(d) $\dfrac{9}{40}$ rad/s

(e) None of the above

24. Given that $\sin^2(t)$ is a periodic function of period $\pi$ and $n$ is an integer, compute the following limit, if it exists.

$$\lim_{n\to\infty} \sin^2(\pi\sqrt{n^{2024} + n^{1012} + 1}).$$

(a) $\dfrac{1}{2}$

(b) 1

(c) 0

(d) Does not exist

(e) None of the above

25. For what value of $x$ does $f(x) = \dfrac{x}{x^2+1}$ attain a relative minimum?

    (a) $x = -1$

    (b) $x = 1$

    (c) $f$ has a relative minimum but it is not given

    (d) $f$ does not have a relative minimum

    (e) None of the above

26. Compute $\lim_{x\to\infty}\left(\dfrac{1}{x^2}\int_1^x t^2 \, \mathrm{d}t\right)$.

    (a) 1

    (b) $\dfrac{1}{3}$

    (c) $-3$

    (d) $\infty$

    (e) None of the above

27. Find the derivative of the function $f(x) = \tan\left(x^5 - 7 + 9\sin^2 x + 9\cos^2 x\right)$.

    (a) $\dfrac{\mathrm{d}f}{\mathrm{d}x} = \sec^2\left(x^5 - 7 + 9\sin^2 x + 9\cos^2 x\right)$

    (b) $\dfrac{\mathrm{d}f}{\mathrm{d}x} = 5x^4 + 9\cos^2 x - 9\sin^2 x$

    (c) $\dfrac{\mathrm{d}f}{\mathrm{d}x} = 5x^4$

    (d) $\dfrac{\mathrm{d}f}{\mathrm{d}x} = 5x^4\sec^2\left(x^5 + 2\right)$

    (e) None of the above

28. **Second Tie-Breaker:** Let $f(x)$ be differentiable function on $\mathbb{R}$ with $f'(x) > 0$ for all $x$ and let $g(x)$ be the inverse of $f(x)$. Then $g'(x)$ is always

(a) negative for all $x$

(b) non-negative for all $x$

(c) positive for all $x$

(d) non-positive for all $x$

(e) None of the above

29. Compute the integral $\int_0^1 \frac{\ln(x+1)}{x^2+1}\, dx$.

(a) $\frac{\pi}{8}$

(b) 0

(c) $\frac{\pi \ln 2}{8}$

(d) $\frac{\ln 2}{8}$

(e) None of the above

30. Compute the integral $\int_4^5 \frac{1}{(x-3)\sqrt{x^2-6x+8}}\, dx$.

(a) $\frac{\pi}{6}$

(b) $-\frac{\pi}{6}$

(c) $\frac{\pi}{3}$

(d) 0

(e) None of the above

31. Find the value of $k$, if any, for which $\int_0^\infty kxe^{-2x}\,\mathrm{d}x = -1$.

(a) $\frac{1}{4}$

(b) $-1$

(c) $-4$

(d) $4$

(e) None of the above

32. Find the derivative of the function $f(x) = x^{\cos x}$, for all $x$ in $(0, \infty)$.

(a) $\frac{\mathrm{d}f}{\mathrm{d}x} = \cos x \cdot x^{\cos x - 1}$

(b) $\frac{\mathrm{d}f}{\mathrm{d}x} = -\cos x \cdot x^{\cos x}$

(c) $\frac{\mathrm{d}f}{\mathrm{d}x} = x^{\cos x}\left[\frac{\cos x}{x} - \sin x \ln x\right]$

(d) $\frac{\mathrm{d}f}{\mathrm{d}x} = x^{\cos x} \ln(\cos x)$

(e) None of the above

33. Let $f(x) = \begin{cases} \left(x^2\right)^{x^2}, & x \neq 0 \\ 1, & x = 0 \end{cases}$. Find $f'(0)$.

(a) $-2$

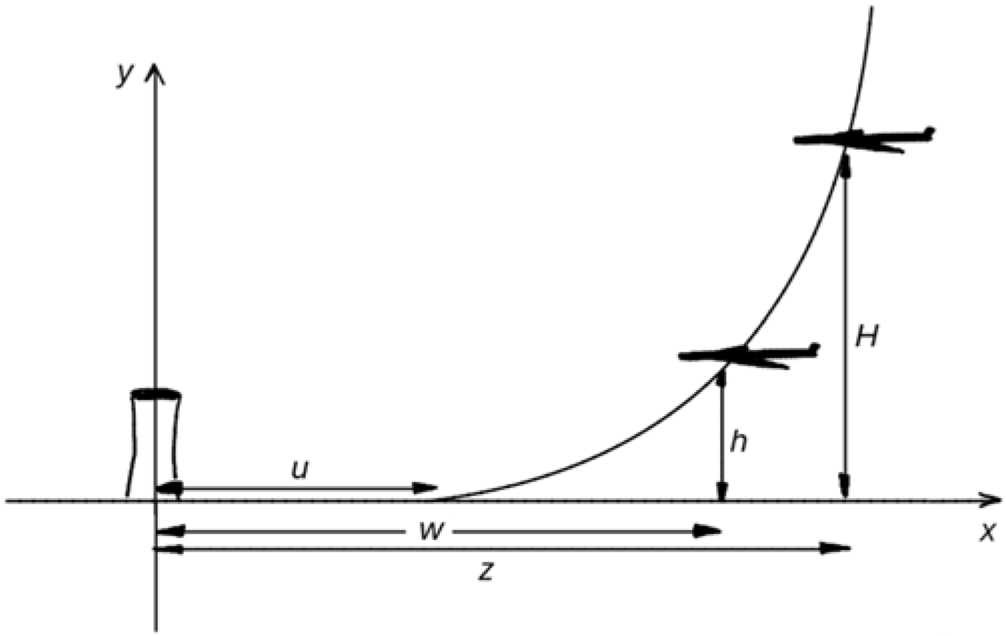
(b) $-1$

(c) $0$

(d) $2$

(e) None of the above

34. A plane getting ready to land moves on the parabola in such a way that when it lands the flight trajectory is tangent to the ground. (See the picture.) In what distance $u$ to the flight control tower will the plane land, if at the distance $z = 9$ km the plane is at the altitude $H = 400$ m, and at the distance $w = 6$ km the plane is at the altitude $h = 100$ m? (1 km = 1000 m)




(a) $u = 4000$ m $= 4$ km

(b) $u = 3000$ m $= 3$ km

(c) $u = 2500$ m $= 2.5$ km

(d) $u = 2000$ m $= 2$ km

(e) None of the above

35. A lighthouse is situated 3 miles from a straight coastline. The rotating light of the lighthouse rotates at the rate of 5 revolutions per minute. Let $P$ be the point at which the beam hits the shore and let $A$ be the point on the shore directly opposite the lighthouse. How fast is the point $P$ moving when $P$ is 4 miles from point $A$?

(a) $\frac{10\pi}{3}$ miles per minute

(b) $\frac{18\pi}{5}$ miles per minute

(c) $\frac{250\pi}{3}$ miles per minute

(d) $\frac{125}{3}$ miles per minute

(e) None of the above

36. Compute $\lim_{x\to 0} \frac{\sqrt[3]{1+x} - \sqrt[3]{1-x}}{x}$.

(a) $\frac{2}{3}$

(b) 0

(c) $\infty$

(d) $\frac{1}{2}$

(e) None of the above

37. Compute the length of the curve $y = \sqrt{x^3}$ for $0 \leq x \leq 4$.

(a) $\frac{8}{27}(10^{3/2} - 1)$

(b) $8(10^{3/2} - 27)$

(c) $\frac{8}{27}$

(d) $\pi$

(e) None of the above

38. **Third Tie-Breaker:** Compute the integral $\int_0^3 \frac{dx}{x-1}$.

(a) $\ln(2)$

(b) $\ln(3)$

(c) $\ln\left(\frac{3}{2}\right)$

(d) Does not exist

(e) None of the above

39. A cubical sponge is absorbing water which causes it to expand. The sponge absorbs water at a rate of 2 cubic inches per minute. How fast is the edge growing when the volume of the sponge is 8 cubic inches?

(a) $\frac{1}{2}$ inch per minute

(b) $\frac{1}{6}$ inch per minute

(c) $\frac{1}{32}$ inch per minute

(d) $\frac{1}{\sqrt[3]{2}}$ inch per minute

(e) None of the above

40. Find the $2025^{\text{th}}$ derivative of the function $y = \sin x$.

(a) $\sin x$

(b) $-\sin x$

(c) $\cos x$

(d) $-\cos x$

(e) None of the above