Gainesville College
Tenth Annual Mathematics Tournament
For Two-Year Colleges
April 3, 2004

Morning Component

Good morning!

Please do NOT open this booklet until given the signal to begin.

There are 40 multiple choice questions. Answer the questions on the electronic grading form by giving the best answer to each question.

The scoring will be done by giving one point for each question answered correctly and zero points for each question answered incorrectly or left blank. Thus, it is to your advantage to answer as many questions as possible, even if you have to guess.

This test was designed to be a CHALLENGE. It is difficult, and you may not have time to complete all questions. Do not worry if you are unable to answer several of the questions. Instead, we hope that you will obtain satisfaction from those questions which you ARE able to answer.

You may write in the test booklet. You may keep your test booklet and any of your scrap papers. Only the electronic grading form will be collected and graded.

Good luck!

Do Not Open Until Signaled.

Gainesville College – Tenth Annual Mathematics Tournament

You may write in this test booklet. Only the electronic form will be graded. Correct answers are awarded one point. Incorrect or blank answers are awarded 0 points.

1. Let f be a one-to-one continuous function such that $f(2)=3$ and $f(5)=7$.

   Given $\int_2^5 f(x)\,dx = 17$, evaluate $\int_3^7 f^{-1}(x)\,dx$.

   a) 10
   b) 11
   c) 12
   d) 13
   e) none of the above

2. Find: $\lim\limits_{x\to\infty} \tan^{-1} x$

   a) $\frac{\pi}{2}$
   b) ∞
   c) $-\infty$
   d) $-\frac{\pi}{2}$
   e) none of the above

3. Find the value of: $\int_1^3 \frac{dy}{y^2-2y+5}$

   a) $\frac{\pi}{8}$
   b) $\ln 8 - \ln 4$
   c) $\frac{\pi}{4}$
   d) $-\frac{1}{8}$
   e) none of the above

4. Let $f(x)$ be a function such that the graph of $f(x)$ is a semicircle with end points $(a,0)$ and $(b,0)$ where $a < b$. Find: $\left|\int_a^b f(x)\,dx\right|$

a) $f(b) - f(a)$

b) $\dfrac{f(b) - f(a)}{b - a}$

c) $(b-a)^2\left(\dfrac{\pi}{8}\right)$

d) $(b-a)\left(\dfrac{\pi}{4}\right)$

e) none of the above

5. Find the area of the largest rectangle that can be inscribed in a semicircle with radius 2.

a) 4
b) 3
c) 2
d) 1
e) none of the above

6. Calculate $\dfrac{d}{dx}\left[\int_{-3}^{x}\left(t^6 + t^5 + t^4 + t^3 + t^2 + t\right)dt\right]$ at $x = 1$.

a) 3
b) 6
c) 12
d) 24
e) none of the above

7. Find the equation(s) of all vertical tangent line(s) to the equation $x^3 + y^2 = xy$.

a) $x = 0$

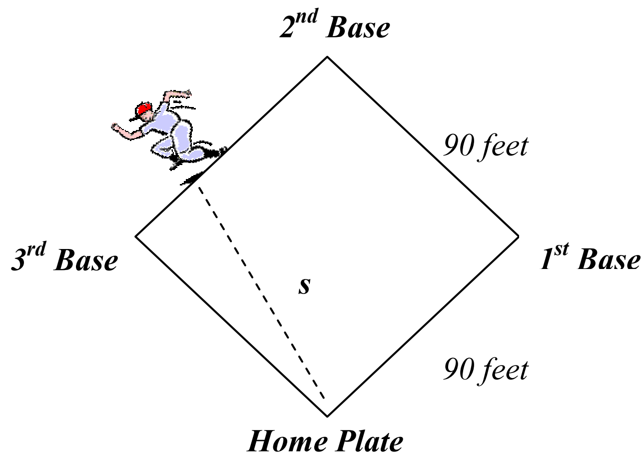
b) $x = \dfrac{1}{8}$

c) $x = \dfrac{1}{4}$

d) more than one vertical tangent line exists

e) none of the above

8. A baseball diamond has the shape of a square with sides 90 feet long. A player running from 2nd base to 3rd base has a constant speed of $25\, ft/s$. At what speed is the player's distance, ***s,*** from home plate changing when the player is 30 feet from 3rd base?



a) $\frac{2\sqrt{10}}{5}\, ft/sec$

b) $5\sqrt{10}\, ft/sec$

c) $\frac{5\sqrt{10}}{2}\, ft/sec$

d) $\frac{5\sqrt{5}}{2}\, ft/sec$

e) none of the above

9. Evaluate the following if it exists: $\lim_{x\to\infty}\left[\ln x - \ln\left(\sqrt{x^2+1}+x\right)\right]$

a) $\frac{1}{2}$

b) $\ln\left(\frac{1}{2}\right)$

c) does not exist

d) 0

e) none of the above

10. Find: $\int_0^{x} \frac{dx}{1+\tan^2 x}$

a) $\frac{1}{2}$

b) $\frac{\pi}{2}$

c) π

d) ∞

e) none of the above

11. Solve the differential equation $\frac{dy}{dx} = \frac{2x}{x^2 - 9}$, finding the solution which passes through the point (0,4).

a) $y = \ln\left|x^2 - 9\right| + 4 - \ln 9$

b) $y = \ln\left|x^2 - 9\right| - 4 + \ln 9$

c) $y = \ln\left|x^2 - 9\right| - 4 - \ln 9$

d) $y = \ln\left|x^2 - 9\right| + 4 + \ln 9$

e) none of the above

12. Use the accompanying figure to calculate: $\int_0^{\pi/2} \sin x\, dx + \int_0^1 \arcsin x\, dx$

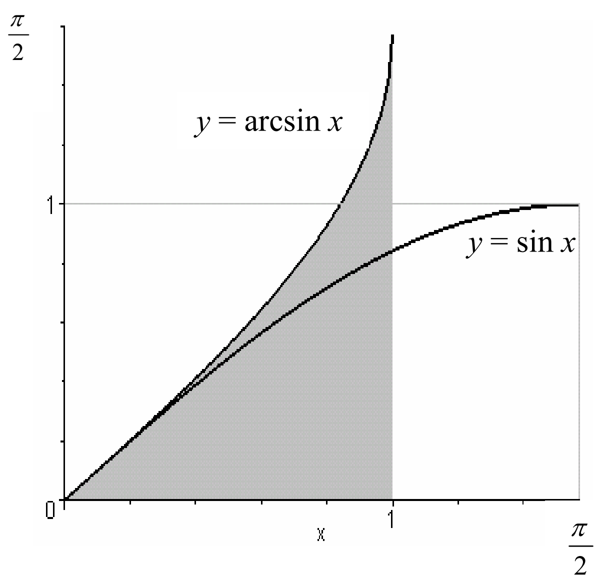
a) $\frac{\pi}{2}$

b) $\frac{\pi}{4}$

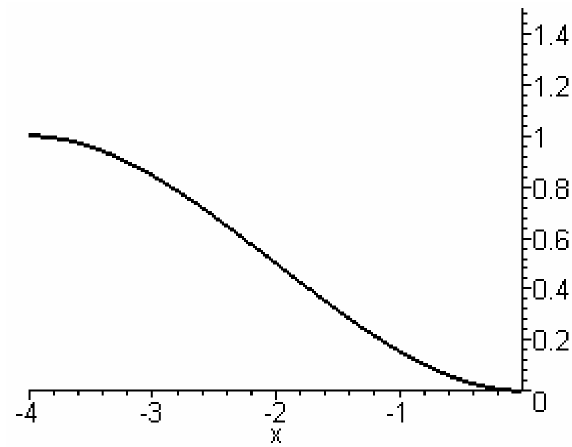
c) 1

d) π

e) none of the above



13. A small aircraft is flying horizontally until it starts its descent from an altitude of 1 mile, 4 miles west of the runway. Find the cubic function $f(x) = ax^3 + bx^2 + cx + d$ on the interval [-4,0] that describes a smooth glide path for the landing, as modeled by the figure below.



a) $f(x) = \frac{1}{32}x^3 + \frac{5}{16}x^2 + \frac{1}{2}x$

b) $f(x) = \frac{1}{16}x^3 + \frac{5}{16}x^2$

c) $f(x) = \frac{1}{32}x^3 + \frac{3}{16}x^2$

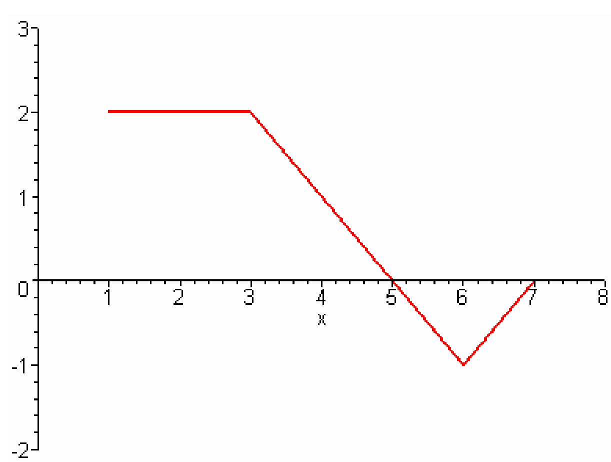
d) $f(x) = \frac{1}{32}x^3 - \frac{3}{4}x^2$

e) none of the above

14. Let $x \sin \pi x = \int_0^{x^2} f(t)\,dt$. Find $f(4)$.

a) 16π
b) 0
c) 4
d) $\frac{\pi}{2}$
e) none of the above

15. The graph of f is given in the figure below. Determine the average value of f on the interval [1,7].



a) $\frac{7}{6}$

b) $\frac{5}{6}$

c) 1

d) $\frac{2}{3}$

e) none of the above

16. If $y = \frac{x^n}{n}$, where n is a positive whole number, what is $\frac{d^n y}{dx^n}$?

a) 1
b) $n - 1$
c) $(n-1)!$
d) $n!$
e) none of the above

17. Find: $\lim_{x \to \pi} \frac{e^{-\pi} - e^{-x}}{\sin x}$

a) $-\infty$
b) $-e^{-\pi}$
c) 0
d) $e^{-\pi}$
e) none of the above

18. Find: $\int_4^5 t \sqrt[3]{t-4}\, dt$

a) $\frac{24}{7}$
b) $-\frac{24}{7}$
c) $\frac{9}{7}$
d) $\frac{7}{24}$
e) none of the above

19. Find: $\int_0^{\infty}\left(1-\frac{e^x}{2+e^x}\right)dx$

a) 0
b) $\ln 2$
c) $\ln 3$
d) ∞
e) none of the above

20. Find the area of the following region: $\begin{cases} y \geq x^2 \\ x^2+y^2 \leq 2 \end{cases}$

a) $\frac{1}{3}+\frac{\pi}{2}$
b) $\frac{\pi}{2}$
c) $2\pi-\frac{4}{3}$
d) $\frac{4}{3}$
e) none of the above

21. Given $f(x)=x^2e^{-x^2}$, find all values of x for which f has a horizontal tangent line.

a) 0
b) 0 or ± 1
c) 0 or -2
d) 0 or ± 2
e) none of the above

22. Given $\int_1^3 f(x)\,dx = 4$ and $\int_1^3 g(x)\,dx = 2$, find $\int_1^3 \left[2f(x)+5g(x)\right] dx$.

a) 54
b) 13
c) 24
d) 18
e) none of the above

23. Assume that *f* is differentiable for all *x*. The sign of f' is as follows:

$$f'(x) > 0 \text{ on } (-\infty, -4)$$
$$f'(x) < 0 \text{ on } (-4, 6)$$
$$f'(x) > 0 \text{ on } (6, \infty)$$

Let $g(x) = f(10-2x)$. Then $g'(5)$ is

a) positive
b) negative
c) zero
d) The function *g* is not differentiable at $x = 5$.
e) none of the above

24. Let $f(x) = \cos(2x)$. Find $f^{(2004)}(0)$, where $f^{(n)}(x)$ denotes the n^{th} derivative of $f(x)$.

a) -2^{2004}
b) 2^{2004}
c) 0
d) 1
e) none of the above

25. Which of the following is true of the behavior of $f(x) = \dfrac{x^3+8}{x^2-4}$ as $x \to 2$?

a) The limit is 0.
b) The limit is 1.
c) The left-hand and the right-hand limits are finite, but not equal.
d) The graph of the function has a vertical asymptote at $x = 2$.
e) none of the above

26. Find: $\displaystyle\int_0^1 (x \ln x)\, dx$

a) -1
b) $-\dfrac{1}{2}$
c) $-\dfrac{1}{4}$
d) 1
e) none of the above

27. Let $f(x) = \begin{cases} 0, & \text{if } x \text{ is rational} \\ 5x, & \text{if } x \text{ is irrational} \end{cases}$. Then

a) f is continuous at every rational number.
b) f is continuous at every irrational number.
c) f is discontinuous everywhere.
d) f is continuous only at $x = 0$.
e) none of the above

28. An entrepreneur is told that to manufacture a newly designed bike, the expenses break down as follows:

30 dollars per bike in parts
20 dollars per bike in labor
10,000 dollars per day in fixed plant costs

Further, she is told that with the conditions of the plant and the nature of the workers, manufacturing 100 bicycles a day should optimize the usefulness of the equipment and the employees. If more or less than 100 bicycles are manufactured, she should expect additional costs of 2 dollars times the squared difference between the number manufactured and 100. To minimize the average cost per bike, how many should she have manufactured each day?

a) fewer than 100
b) exactly 100
c) more than 100 but not more than 110
d) more than 110 but not more than 120
e) more than 120

29. Suppose *f* and *g* are differentiable, and that $f(x) = x^2 g(x)$. If $g(2) = 3$ and $g'(2) = -2$, then find $f'(2)$.

a) -8
b) 20
c) 12
d) 4
e) none of the above

30. Given $\int_{-1}^{1} f(x)\,dx = 5$ and $\int_{0}^{1} f(x)\,dx = 7$, calculate $\int_{0}^{-1} 3f(x)\,dx$.

a) 3
b) 6
c) -6
d) -3
e) none of the above

31. Let $f(x)=\dfrac{\sqrt{x^2+kx+1}}{x^2-k}$. Identify the interval(s) of all possible values of k for which f is continuously defined over $(-\infty,\infty)$.

a) $(-\infty,0)$
b) $(-\infty,0)\cup(0,\infty)$
c) $(0,2]$
d) $[-2,0)$
e) none of the above

32. If $f(x)=\begin{cases}\dfrac{|x|}{x}, & \text{for } x\neq 0\\ 0, & \text{for } x=0\end{cases}$, then $\int_{-1}^{1} f(x)\,dx$ is equal to:

a) -2
b) 0
c) 2
d) undefined
e) none of the above

33. Find: $\lim\limits_{x\to 0}\dfrac{4\left(e^{2x}-1\right)}{e^x-1}$

a) 8
b) 4
c) e
d) 0
e) none of the above

34. If $\int_0^b \tan x \, dx = 2$, then b could equal

a) $\operatorname{arccos}(2)$

b) $\operatorname{arccos}(2e)$

c) $\operatorname{arcsec}(2)$

d) $\operatorname{arcsec}^2(e)$

e) none of the above

35. Determine the values of k such that the function $f(x) = kx + \cos x$ is a one-to-one function.

a) k is positive
b) $-1 \le k \le 1$
c) k is negative
d) $k \le -1$ or $k \ge 1$
e) none of the above

36. Find two positive numbers in which the sum of the first and twice the second is 100 and the product is a maximum.

a) 15, 70
b) 10, 80
c) 24, 52
d) 20, 60
e) none of the above

37. Let $y = w^4$, $w = 2 + 3u$, and $u = \ln x$. Express y in terms of x and find $\frac{dy}{dx}$.

a) $y = (3\ln x)^4$; $\frac{dy}{dx} = \frac{12(2+3\ln x)^3}{x}$

b) $y = (2+3\ln x)^4$; $\frac{dy}{dx} = \frac{12(2+3\ln x)^3}{x}$

c) $y = (2+3\ln x)^4$; $\frac{dy}{dx} = \frac{6(2+3\ln x)^3}{x}$

d) $y = (3\ln x)^4$; $\frac{dy}{dx} = \frac{6(2+3\ln x)^3}{x}$

e) none of the above

38. At what rate is the distance between the tip of the hour hand and the 12 o'clock mark changing when the hour hand points to 4 o'clock? The length of the hour hand is 20 *cm*.

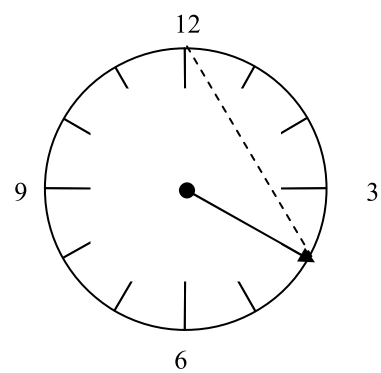
a) $\frac{\sqrt{3}}{5}\pi$ *cm/hr*

b) $\frac{5}{6}\pi$ *cm/hr*

c) $\frac{400}{3}\pi$ *cm/hr*

d) $\frac{5}{3}\pi$ *cm/hr*

e) none of the above



39. What is the greatest number of relative extrema that a polynomial function of the form $y = x^n - nx^{n-2}$ can have, where n is a whole number and $n \geq 2$?

a) 0
b) 2
c) 3
d) $n - 2$
e) none of the above

40. For $f(x) = \sin\left[\cos\left(2x - \frac{\pi}{2}\right)\right]$, find $f'\left(-\frac{\pi}{4}\right)$.

a) -1
b) 1
c) -2
d) 0
e) none of the above