Gainesville College
Eleventh Annual Mathematics Tournament
For Two-Year Colleges
April 2, 2005

Solutions for the Afternoon Team Competition

Round 1: Let A= the area of the figure, A_S = the area of the circular sector of the circle of radius 2 subtended by a central angle of 60°, and A_T = the area of the equilateral triangle of side length 2. Then $A = 3A_S - 2A_T$.

$A_S = \frac{1}{6}\pi(2)^2$, since the sector is one sixth of the circle, and $A_T = \frac{\sqrt{3}}{4}(2)^2$.

So $A = 3\left(\frac{2}{3}\pi\right) - 2\sqrt{3} = 2\left(\pi - \sqrt{3}\right)$

Round 2: Let x be the time pump A can empty the pool by itself and y be the time pump B can empty the pool by itself. Then,

$$\left(\frac{1}{x}\right)5 + \left(\frac{1}{y}\right)3 = 1 \text{ and } \left(\frac{1}{x}\right)3 + \left(\frac{1}{y}\right)6 = 1$$

If we solve the system of equations, we will obtain $x = 7$ and $y = 10.5$.

Now let t be the time for both pumps working together to empty the pool. Thus,

$$\left(\frac{1}{7}\right)t + \left(\frac{1}{10.5}\right)t = 1 \Rightarrow \left(\frac{1}{7} + \frac{2}{21}\right)t = 1 \Rightarrow \left(\frac{5}{21}\right)t = 1$$

Therefore, $t = \frac{21}{5} = 4.2$ *hours*.

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Round 3: $x+y+z=2 \Rightarrow (x+y+z)^2=(2)^2$. So,

$$x^2+y^2+z^2+2xy+2xz+2yz=4 \Rightarrow x^2+y^2+z^2+2(xy+xz+yz)=4.$$

Since $xy+xz+yz=1$, we have

$$x^2+y^2+z^2+2=4 \Rightarrow x^2+y^2+z^2=2.$$

Round 4: $\frac{1}{7}=0.\overline{142857}$

Since 2005 = 6(334)+1, the 2005th digit of the decimal representation of $\frac{1}{7}$ is the same as the first digit after the decimal point, which is 1.

Round 5: Let x be the number of five-dollar bills and y be the number of one-dollar bills Michael originally had. Then we have

$$\left(\frac{2}{3}\right)(5x+y)=5y+x$$

Solving for y, we get $y=7\left(\frac{x}{13}\right)$. Since y is an integer, then x must be divisible by 13. So, we get the following pairs, and the total sum of money.

x	y	$Total = 5x+y$
13	7	72
26	14	144
39	21	216

Since he had between \$140 and \$150, it must be \$144.

Round 6: The total number of crossing points is $10 \times 7 = 70$. The total number of ordered pairs of points is $70 \times 69 = 4830$. The number of ordered horizontal pairs is $7 \times 6 \times 10 = 420$. The number of ordered vertical pairs is $10 \times 9 \times 7 = 630$. The number of ordered non-vertical, non-horizontal pairs is $4830 - 420 - 630 = 3780$.

One rectangle can be defined by giving an ordered non-vertical, non-horizontal pair of points that make the opposite corners. However, the same rectangle is given by 4 different such pairs.

Therefore, the number of rectangles is $\frac{3780}{4} = 945$.

Round 7: Since $f\left(g\left(x\right)\right) = 3 + 2\sqrt{x} + x = 2 + \left(1 + 2\sqrt{x} + x\right) = 2 + \left(1 + \sqrt{x}\right)^2$, we can have $f\left(x\right) = 2 + x^2$.

Round 8: $\frac{b_1}{b_2} = \frac{1}{13} \Rightarrow b_2 = 13b_1 = 13 \times .5 = 6.5\ ft$.

$$A = 28\ ft^2 \text{ and } A = \frac{\left(b_1 + b_2\right)h}{2} = \frac{\left(7\ ft\right)h}{2} \Rightarrow \frac{7h}{2} = 28 \Rightarrow h = 8\ ft$$

$$\left(b_2 - b_1\right)^2 + h^2 = x^2 \Rightarrow \left(6\ ft\right)^2 + \left(8\ ft\right)^2 = x^2 \Rightarrow x^2 = 100\ ft^2 \Rightarrow x = 10\ ft$$

Round 9: To number the first 9 pages took 9 digits.

To number the pages between 10 and 99 took $2 \times 90 = 180$ digits.

To number the pages between 100 and 999 took $3 \times 900 = 2700$ digits.

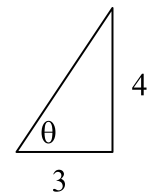
So pages 1 to 999 took $9 + 180 + 2700 = 2889$ digits.

The remaining number of digits is $3293 - 2889 = 404$.

There are 4 digits per page at this time, so we have an extra 101 pages.

The total number of pages in the book is then $999 + 101 = 1100$.

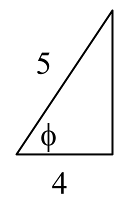
Round 10: Let $\theta = tan^{-1}\left(\frac{4}{3}\right)$. Then the triangle at the right could be drawn.



By the Pythagorean Theorem, the length of the hypotenuse is 5, and $sin\,\theta = \frac{4}{5}$. Therefore,

$$cos\left(sin^{-1}\left(tan\left(cos^{-1}\left(sin\left(tan^{-1}\left(\frac{4}{3}\right)\right)\right)\right)\right)\right) = cos\left(sin^{-1}\left(tan\left(cos^{-1}\left(\frac{4}{5}\right)\right)\right)\right)$$

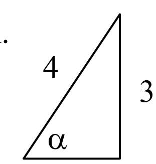
Now, let $\phi = cos^{-1}\left(\frac{4}{5}\right)$. Then the triangle at the right could be drawn.



By the Pythagorean Theorem, the length of the missing side is 3, and $tan\,\phi = \frac{3}{4}$. Then,

$$cos\left(sin^{-1}\left(tan\left(cos^{-1}\left(\frac{4}{5}\right)\right)\right)\right) = cos\left(sin^{-1}\left(\frac{3}{4}\right)\right)$$

Finally, let $\alpha = sin^{-1}\left(\frac{3}{4}\right)$. Then the triangle at the right could be drawn.



By the Pythagorean Theorem, the length of the missing side is $\sqrt{7}$, and $cos\,\alpha = \frac{\sqrt{7}}{4}$. Thus,

$$cos\left(sin^{-1}\left(\frac{3}{4}\right)\right) = \frac{\sqrt{7}}{4}$$