Gainesville College
Eleventh Annual Mathematics Tournament
For Two-Year Colleges
April 2, 2005

Morning Component

Good morning!

Please do NOT open this booklet until given the signal to begin.

There are 40 multiple choice questions. Answer the questions on the electronic grading form by giving the best answer to each question.

The scoring will be done by giving one point for each question answered correctly and zero points for each question answered incorrectly or left blank. Thus, it is to your advantage to answer as many questions as possible, even if you have to guess. If there is a tie, question number 23 will be used again as a tie-breaker.

This test was designed to be a CHALLENGE. It is difficult, and you may not have time to complete all questions. Do not worry if you are unable to answer several of the questions. Instead, we hope that you will obtain satisfaction from those questions which you ARE able to answer.

You may write in the test booklet. You may keep your test booklet and any of your scrap papers. Only the electronic grading form will be collected and graded.

Good luck!

Do Not Open Until Signaled.

Gainesville College – Eleventh Annual Mathematics Tournament

You may write in this test booklet. Only the electronic form will be graded. Correct answers are awarded one point. Incorrect or blank answers are awarded 0 points.

1. Find $\int \frac{\log_2 x}{x}\,dx$.

   a) $\frac{1}{2}(\log_2 x)^2 + C$
   b) $\frac{1}{2}(\log_2 x)(\ln x) + C$
   c) $\frac{1}{\ln 2}(\ln x)^2 + C$
   d) $\frac{1}{\ln 2}(\log_2 x)(\ln x)\ln x + C$
   e) none of the above

2. A general solution for the differential equation $\left(1+x^2\right)\frac{dy}{dx} - 2xy = 0$ is

   a) $y = \frac{C}{1+x^2}, C \geq 0$
   b) $y = C\left(1+x^2\right), C \geq 0$
   c) $y = \left(1+x^2\right) + C, C \geq 0$
   d) no solution exists
   e) none of the above

3. Let $f(x)$ be a one-to-one continuous function such that $f(1)=4$ and $f(6)=2$. Assume $\int_1^6 f(x)\,dx = 15$. Calculate $\int_2^4 f^{-1}(x)\,dx$.

a) 5
b) 6
c) 7
d) 8
e) none of the above

4. A function $f(x)$ has a relative maximum value at $x=c$. Which of the following must be true?

a) $f'(c)=0$
b) f' changes from positive to negative at c
c) $f''(c)<0$
d) all of the above
e) none of the above

5. How many real solutions does the equation $x^7+x^5+x^3+1=0$ have?

a) None
b) One
c) Three
d) Seven
e) none of the above

6. If $f(x)=\dfrac{(x-1)^3(x+2)^4}{(x+1)^3}$, what is the equation of the tangent line at $x=0$?

a) $y=-16x+64$
b) $y=-16x$
c) $y=64x$
d) $y=64x-16$
e) none of the above

7. The length of a rectangle is 16 inches and the width is 13 inches. If the area A is increasing at 1 in^2/min, at what rate must the width be changing so that the length is increasing at 5 in/min?

a) $-4\ in/min$
b) $4\ in/min$
c) $\frac{1}{5}\ in/min$
d) $-\frac{1}{5}\ in/min$
e) none of the above

8. Find the maximum value of the function $f(x) = -\left(e^{\frac{x}{2}} + e^{-\frac{x}{2}}\right)$.

a) -2
b) 0
c) -1
d) $-e$
e) none of the above

9. Find $\frac{d^2y}{dx^2}$ if $x^2 - y^4 = 6$.

a) $\frac{-3}{16y^7}$
b) $\frac{y}{3x}$
c) $\frac{2y-6x}{4y^4}$
d) $\frac{2y^4-3x^2}{4y^7}$
e) none of the above

10. Find $\lim_{y\to 2}\left[\frac{1}{y-2}\left(\frac{1}{x+y-2}-\frac{1}{x}\right)\right]$.

a) 0
b) $\ln x$
c) $-\frac{1}{x^2}$
d) ∞
e) none of the above

11. Evaluate $\int_0^{\frac{\pi}{2}} \frac{\cos x}{\sqrt{\sin x}}\,dx$.

a) 0
b) 1
c) 2
d) 3
e) none of the above

12. Find the differential dy of $y=\frac{e}{e^x}$ when $x=1$.

a) $1\,dx$
b) $0\,dx$
c) $-1\,dx$
d) $e\,dx$
e) none of the above

13. Tell where the function given by $f(x)=\sqrt{\frac{1-x^2}{4-x^2}}$ is continuous.

a) $[-1,1]$
b) $(-\infty,-2)\cup(2,\infty)$
c) $(-\infty,-2)\cup(-1,1)\cup(2,\infty)$
d) $(-\infty,-2)\cup[-1,1]\cup(2,\infty)$
e) none of the above

14. Find the volume V of the solid of revolution formed by revolving the region bounded by $y = \frac{1}{x}$, $y = 0$, $x = 1$, and $x = e$ about the y-axis.

a) 2π
b) 1
c) $2\pi(e-1)$
d) $e-1$
e) none of the above

15. Find the point of inflection of $r(x) = \frac{x-2}{(x+1)^2}$.

a) $(2,0)$
b) $\left(5,\frac{1}{12}\right)$
c) $\left(8,\frac{2}{27}\right)$
d) $\left(4,\frac{2}{25}\right)$
e) none of the above

16. If f is continuous on $[0,2]$, differentiable on $(0,2)$, $f(0)=2$, $f(2)=8$, and $f'(x) \le 3$ for all x in $(0,2)$, find $f(1)$.

a) 3
b) 1
c) 10
d) There is not enough information.
e) none of the above

17. Find all values of k so that $y = kx$ is tangent to $y = x^2 + k$.

a) 0
b) 0 and 2
c) 0 and 4
d) 0 and $\frac{1}{4}$
e) none of the above

18. If x and y are real numbers such that $x^2 + y^2 = 8$, what is the maximum possible value of $x - y$?

a) 2
b) $\sqrt{2}$
c) $\dfrac{\sqrt{2}}{2}$
d) 4
e) none of the above

19. If $y = n(x-1)^n$, where n is a positive integer, what is $\dfrac{d^n y}{dx^n}$?

a) $(n^2)!$
b) $(n-1)!\,n^2$
c) 0
d) $n!$
e) none of the above

20. Evaluate $\lim_{x\to 0}(1-3x)^{1/x}$.

a) e^3
b) e^{-3}
c) 1
d) ∞
e) none of the above

21. Evaluate: $\int_0^4 \sqrt{16-x^2}\, dx$

a) 16π
b) 8π
c) 4π
d) 2π
e) none of the above

22. Determine which function would produce the greatest area between the function and $g(x) = 0$ from $x = 1$ to $x = 100$.

a) $f(x) = x^{10}$
b) $f(x) = 10^x$
c) $f(x) = 10x$
d) $f(x) = \log_{10}(x^{10})$
e) none of the above

Reminder
Question 23 will be used again as a tie-breaker, if necessary.

23. Consider the particle traveling clockwise on the elliptical path $\frac{x^2}{100} + \frac{y^2}{25} = 1$. The particle leaves the orbit at the point $(-8, 3)$ and travels in a straight line tangent to the ellipse. At what point will the particle cross the y-axis?

a) $\left(0, \frac{25}{3}\right)$
b) $\left(0, -\frac{25}{3}\right)$
c) $(0, 9)$
d) $\left(0, \frac{7}{3}\right)$
e) none of the above

24. If the tangent line to $y = f(x)$ at (a, b) has slope $m \neq 0$, then what slope does the tangent line to $y = f^{-1}(x)$ at (b, a) have?

a) m
b) $-m$
c) $\frac{1}{m}$
d) $-\frac{1}{m}$
e) none of the above

25. Let f be continuous on $[-1,3]$ and differentiable on $(-1,3)$, with $f(-1)=5$ and $f(3)=10$. Then there must be a number k in $(-1,3)$ such that

a) $f'(k)=\frac{5}{4}$
b) $f(k)=10$
c) $f''(k)\geq 0$
d) $f'(k)=0$
e) none of the above

26. Two lines pass through the point $(3,0)$ and are tangent to the parabola $y=x^2$. One of the lines is the x-axis itself. Find an equation for the other line.

a) $x=3$
b) $y=12x-36$
c) $y=6x-18$
d) $y=3x-9$
e) none of the above

27. If $f(x)=x^4-cx$, then the minimum value of $f(x)$ is

a) $f(c)$
b) $f\left(\sqrt[3]{\frac{c}{4}}\right)$
c) no minimum exists
d) $f\left(\sqrt[3]{c}\right)$
e) none of the above

28. Find the 151$^{\text{st}}$ derivative of $f(x)=\sin(-x)$.

a) $-\cos(-x)$
b) $\sin(-x)$
c) $\cos x$
d) $\sin x$
e) none of the above

29. Find all critical numbers of the greatest integer function $f(x)=[\![x]\!]$.

a) all integers
b) all real numbers except integers
c) all real numbers
d) no critical numbers
e) none of the above

30. Evaluate: $\lim_{x\to 1} \dfrac{x-1}{\ln\left(x^2\right)}$

a) $\dfrac{1}{2}$
b) ∞
c) 1
d) 0
e) none of the above

31. Evaluate: $\int_{-1}^{2} \left|e^x-1\right| dx$

a) $e^2-e^{-1}-3$
b) $e^2+e^{-1}+3$
c) $e^2-e^{-1}+3$
d) $e^2+e^{-1}-3$
e) none of the above

32. Which of the following definite integrals has a positive value?

a) $\int_0^{\frac{2\pi}{3}} \sin(3x+\pi)\,dx$

b) $\int_{\frac{2\pi}{3}}^{0} \sin(3x+\pi)\,dx$

c) $\int_{\frac{-3\pi}{2}}^{0} \sin(3x+\pi)\,dx$

d) $\int_0^{\frac{-3\pi}{2}} \sin(3x+\pi)\,dx$

e) none of the above

33. Solve the differential equation $f''(x)=\cos x$, $f'\left(\frac{3\pi}{2}\right)=e$, $f(0)=-1$.

a) $\sin x-(e+1)x$

b) $\sin x+(e+1)x$

c) $(e+1)x+\cos x$

d) $(e+1)x-\cos x$

e) none of the above

34. Find: $\int x^2 \ln x\,dx$

a) $\frac{1}{3}x^3\ln x-\frac{1}{9}x^3+C$

b) $2x\ln x+x+C$

c) $\frac{1}{3}x^3\ln x-\frac{1}{3}x^2+C$

d) $\frac{1}{3}x^3\ln x+x+C$

e) none of the above

35. Find: $\int \frac{1}{x^2\sqrt{16-x^2}}\,dx$

a) $-\frac{1}{4}\operatorname{arcsec}\left(\frac{x}{4}\right)+C$

b) $\frac{1}{4}\operatorname{arcsec}\left(\frac{x}{4}\right)+C$

c) $-\frac{\sqrt{16-x^2}}{16x}+C$

d) $\frac{\sqrt{16-x^2}}{16x}+C$

e) none of the above

36. The derivative of $f(x)=5x^x$ is

a) $5xx^{x-1}$

b) $5x^x \ln x$

c) $\frac{5x^x}{\ln x}$

d) $5x^x(1+\ln x)$

e) none of the above

37. How much work is done by a colony of ants in building a conical ant hill with height and diameter of the base both 1 *ft*, using sand initially at ground level and with a density of 150 *lb/ft*^{3}*?*

a) $\frac{75}{8}\pi$ *ft-lb*

b) $\frac{25}{2}\pi$ *ft-lb*

c) $\frac{25}{8}\pi$ *ft-lb*

d) 25π *ft-lb*

e) none of the above

38. Air is escaping from a spherical balloon at the constant rate of 200π *cm*^{3}/*s*. What is the radius of the balloon when its radius is decreasing at 2 *cm/s*?

a) 5 *cm*
b) $5\sqrt{2}$ *cm*
c) 10 *cm*
d) 12.5 *cm*
e) none of the above

39. Find the derivative of $f(x) = \ln(\ln(x))$.

a) $f'(x) = \dfrac{1}{x \ln(x)}$
b) $f'(x) = \dfrac{1}{\ln(x)}$
c) $f'(x) = \dfrac{2\ln(x)}{x}$
d) $f'(x) = \dfrac{1}{x}$
e) none of the above

40. Calculate $\dfrac{d}{dx}\left[\int_x^5 \cos^7 t \, dt\right]$ at $x = 0$.

a) $-\dfrac{1}{2}$
b) $\dfrac{1}{2}$
c) 1
d) -1
e) none of the above