Gainesville State College
Thirteenth Annual Mathematics Tournament
April 14, 2007

Solutions for the Afternoon Team Competition

Round 1

Let x represent the total number of coins the man has.

number of dimes = $\frac{1}{4}x$

number of nickels = $\frac{1}{2}x$

number of quarters = $x-\frac{1}{4}x-\frac{1}{2}x=\frac{4}{4}x-\frac{1}{4}x-\frac{2}{4}x=\frac{1}{4}x$

Now, we have

$$.90=.25\times(\textit{number of quarters})+.10\times(\textit{number of dimes})+.05\times(\textit{number of nickels})$$

$$.90=.25\left(\frac{1}{4}x\right)+.10\left(\frac{1}{4}x\right)+.05\left(\frac{1}{2}x\right)\ \Rightarrow\ .90=\frac{.25x}{4}+\frac{.10x}{4}+\frac{.05x}{2}$$

$$\Rightarrow\ .90=\frac{.25x}{4}+\frac{.10x}{4}+\frac{.10x}{4}\ \Rightarrow\ .90=\frac{.45x}{4}$$

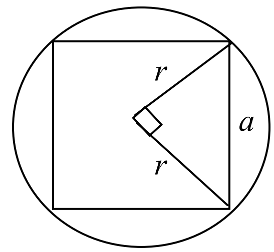
$$\Rightarrow\ 4(.90)=4\left(\frac{.45x}{4}\right)\ \Rightarrow\ 3.6=.45x\ \Rightarrow\ \frac{3.6}{.45}=\frac{.45x}{.45}\ \Rightarrow\ x=8$$

Thus, the number of nickels = $\frac{1}{2}(8)=4$, the number of dimes = $\frac{1}{4}(8)=2$, and the number of quarters = $\frac{1}{4}(8)=2$.

Round 2

The factors must have some order of -2, -1, 1, and 2. Thus, the numbers are 5, 6, 8, and 9. Therefore, the sum is 28.

Round 3



Let r be the radius of the circle and a be the length of a side of the square. Since the area of the circle equals 12.5π square units, we obtain

$$\pi r^2 = 12.5\pi \quad \Rightarrow \quad r^2 = 12.5$$

Now, using the Pythagorean Theorem for the triangle marked in the picture, we obtain

$$a^2 = r^2 + r^2 = 2r^2 = 2 \times 12.5 = 25.$$

So $a = 5$ and the perimeter = $4a = 4 \times 5 = 20$ units.

Round 4

To determine the number of diagonals in an n-gon, start with counting the total number of line segments that connect two vertices, including the sides, and then subtract n from the result to take care of the sides. Starting at a given vertex, there are $n-1$ lines to be drawn to the other vertices. The next vertex already has a line drawn to the first vertex, so only $n-2$ line segments can be drawn. The third vertex can be connected to only $n-3$ vertices, and so on, until we reach the last vertex, which already has a line segment to every other vertex. The total number of line segments drawn is

$$(n-1)+(n-2)+(n-3)+\ldots+1+0=\frac{n(n-1)}{2}$$

Subtracting n gives the number of diagonals of a regular n-gon: $\frac{n(n-1)}{2} - n = \frac{n(n-3)}{2}$.

In our case, we have $\frac{n(n-3)}{2} = 44$. So n must be 11.

Round 5

The difference of consecutive squares is an odd number because $(x+1)^2 - x^2 = 2x+1$.

Therefore, the 964 must be the last three digits of the larger square and the last three digits of the smaller square must be either 469 or 649.

Since the rest of the digits are identical, the difference of squares must be either 964 – 469 = 495 or 964 – 649 = 315.

So either $2x+1=495$ or $2x+1=315$. Hence, either $x=247$ or $x=157$.

$247^2 = 61,009$ and $248^2 = 61,504$ do not work.

But $157^2 = 24,649$ and $158^2 = 24,964$ do work.

Therefore, the consecutive squares are 24,649 and 24,964.

Round 6

$\arctan x + \arctan b = 45^\circ \Rightarrow \arctan x = 45^\circ - \arctan b$.

So, $x = \tan\left(45^\circ - \arctan b\right) = \dfrac{\tan 45^\circ - \tan(\arctan b)}{1 + \tan 45^\circ \tan(\arctan b)} = \dfrac{1-b}{1+b}$.

Round 7

Let $z = \dfrac{1}{x+3}$. Then $x = \dfrac{1-3z}{z}$.

So, $f(z) = f\left(\dfrac{1}{x+3}\right) = \dfrac{1}{2-5x} = \dfrac{1}{2-5\left(\dfrac{1-3z}{z}\right)} = \dfrac{z}{2z-5+15z} = \dfrac{z}{17z-5}$.

Therefore, $f(x) = \dfrac{x}{17x-5}$.

Round 8

$1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + 4 \cdot 5 + \ldots + 98 \cdot 99 + 99 \cdot 100$

$$= (2-1) \cdot 2 + (3-1) \cdot 3 + (4-1) \cdot 4 + \ldots + (99-1) \cdot 99 + (100-1) \cdot 100$$

$$= 2 \cdot 2 - 2 + 3 \cdot 3 - 3 + 4 \cdot 4 - 4 + \ldots + 99 \cdot 99 - 99 + 100 \cdot 100 - 100$$

$$= \left(2^2 + 3^2 + 4^2 + \ldots + 99^2 + 100^2\right) - \left(2 + 3 + 4 + \ldots 99 + 100\right)$$

$$= \left(1^2 + 2^2 + 3^2 + 4^2 + \ldots + 99^2 + 100^2\right) - \left(1 + 2 + 3 + 4 + \ldots 99 + 100\right)$$

$$= \frac{(100)(101)(201)}{6} - \frac{(100)(101)}{2} = 333{,}300$$

Round 9

The speed of Pump A is $\frac{1}{8}$ pool/hour. The speed of Pump B is $\frac{1}{7}$ pool/hour. The combined speed of Pump A and Pump B is $\left(\frac{1}{8}\right)(0.7) + \left(\frac{1}{7}\right)(0.8) = \frac{7}{80} + \frac{8}{70}$ pool/hour. So it will take $1 \div \left(\frac{7}{80} + \frac{8}{70}\right) = 4.955752212$ hours $\approx$ 4 hours and 57 minutes.

Round 10

When the two shaded regions have the same area, the triangle has the same area as the quarter circle. So,

$$\frac{1}{2} \tan \theta = \frac{1}{4} \pi$$

$$\tan \theta = \frac{\pi}{2}$$

$$\theta = \arctan\left(\frac{\pi}{2}\right)$$

$$\theta \approx 1.004$$