Gainesville State College
Fourteenth Annual Mathematics Tournament
April 12, 2008

Morning Component

Good morning!

Please do NOT open this booklet until given the signal to begin.

There are 40 multiple choice questions. Answer the questions on the electronic grading form by giving the best answer to each question.

The scoring will be done by giving one point for each question answered correctly and zero points for each question answered incorrectly or left blank. Thus, it is to your advantage to answer as many questions as possible, even if you have to guess. If there is a tie, question number 12 will be used again as a tie-breaker.

This test was designed to be a CHALLENGE. It is difficult, and you may not have time to complete all questions. Do not worry if you are unable to answer several of the questions. Instead, we hope that you will obtain satisfaction from those questions which you ARE able to answer.

You may write in the test booklet. You may keep your test booklet and any of your scrap papers. Only the electronic grading form will be collected and graded.

Good luck!

Do Not Open Until Signaled.

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minsu.kim@ung.edu or call 678-717-3546.

Fourteenth Annual Gainesville State College Mathematics Tournament

You may write in this test booklet. Only the electronic form will be graded. Correct answers are awarded one point. Incorrect or blank answers are awarded 0 points.

1. At what rate is the area of an equilateral triangle increasing if its perimeter is 18 cm and its perimeter is increasing at 1cm/s?

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a) $\frac{\sqrt{3}}{2}$ cm^2/s

b) $\sqrt{3}$ cm^2/s

c) $2\sqrt{3}$ cm^2/s

d) $4\sqrt{3}$ cm^2/s

e) none of the above

2. Find the length of the curve $y = \frac{1}{2}\left(e^x + e^{-x}\right)$, $0 \le x \le 2$.

a) $2e^2 + \frac{1}{2e^2}$

b) $\frac{e^2}{2} + \frac{1}{e^2}$

c) $\frac{e^2}{2} - \frac{1}{2e^2}$

d) $e^2 - \frac{1}{e^2}$

e) none of the above

3. Let $f(x)=\begin{cases}\dfrac{|x-3|}{x-3}, & x\neq 3\\ 0, & x=3\end{cases}$. Find $\lim\limits_{x\to 3^+} f(x)$.

a) 1

b) -1

c) ∞

d) 3

e) none of the above

4. Find $\dfrac{dy}{dx}$ for $y=x^3\sqrt{x+1}$.

a) $\dfrac{3x^2}{2\sqrt{x+1}}$

b) $\dfrac{x^2(7x+6)}{2\sqrt{x+1}}$

c) $3x^2\sqrt{x+1}$

d) $\dfrac{7x^3+x^2}{2\sqrt{x+1}}$

e) none of the above

5. Let $f(x)$ be a continuous even function over the interval $(-\infty,\infty)$. Given $\int_{-4}^{4} f(x)\,dx=18$, $\int_{-2}^{3} f(x)\,dx=12$, and $\int_{2}^{3} f(x)\,dx=2$. What is $\int_{3}^{4} f(x)\,dx$?

a) 1

b) 2

c) 3

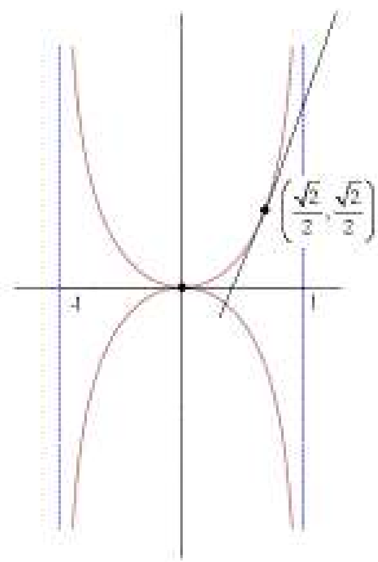
d) 4

e) none of the above

6. Find the value of t such that $f(t)=\begin{cases}-t^2-t+2 & \text{for } t<1\\ t-1 & \text{for } t\ge 1\end{cases}$ is the largest on the closed interval $[-2,4]$.

a) $t=0$
b) $t=2$
c) $t=4$
d) $t=-\dfrac{1}{2}$
e) none of the above

7. Find the slope of the tangent line to the graph given by $x^2(x^2+y^2)=y^2$ at the point $\left(\dfrac{\sqrt{2}}{2},\dfrac{\sqrt{2}}{2}\right)$. This graph is called the Kappa curve because it resembles the Greek letter Kappa, κ.



a) 1
b) $\dfrac{\sqrt{2}}{2}$
c) 2
d) 3
e) none of the above

8. Find the length of the arc of the parabola $y=x^2$ from $x=0$ to $x=1$.

a) $\dfrac{\sqrt{5}}{2}+\dfrac{\ln\left(2+\sqrt{5}\right)}{4}$
b) $\dfrac{\sqrt{5}}{2}-\dfrac{\ln\left(2+\sqrt{5}\right)}{4}$
c) $\dfrac{\sqrt{5}}{2}+\dfrac{\ln\left(\sqrt{5}-2\right)}{4}$
d) $\dfrac{\sqrt{5}}{2}+\dfrac{\ln\left(2+\sqrt{5}\right)}{2}$
e) none of the above

9. Find $\lim_{x\to e}\left(\frac{\ln(\ln x)}{\ln x}\right)$.

a) ln
b) 0
c) 1
d) e
e) none of the above

10. Find the point of inflection for the cubic polynomial $p(x) = ax^3 + bx^2 + cx + d$.

a) $x_0 = -\frac{b}{3a},\ y_0 = \frac{2b^3}{27a^2} - \frac{bc}{3a} + d$

b) $x_0 = -\frac{b}{2a},\ y_0 = \frac{b^3}{9a^2} - \frac{2bc}{3a} + d$

c) $x_0 = -\frac{b}{2a},\ y_0 = \frac{b^2}{8a^2} - \frac{bc}{2a} + d$

d) $x_0 = -\frac{b}{3a},\ y_0 = \frac{b^3c}{27a^3} + \frac{bc}{3a} + d$

e) none of the above

11. Determine the indefinite integral $\int \cos^{-1}(2x)\,dx$.

a) $x\cos^{-1}(2x) - \frac{1}{2}\sqrt{1-4x^2} + C$

b) $2x\cos^{-1}(2x) - \sqrt{1-4x^2} + C$

c) $-x\cos^{-1}(2x) + \frac{1}{2}\sqrt{1-4x^2} + C$

d) $-2x\cos^{-1}(2x) + \sqrt{1-4x^2} + C$

e) none of the above

Reminder
Question 12 will be used again as a tie-breaker, if necessary.

12. What is the smallest slope that the tangent line to the curve $y = x^5 + 2x$ can have?

a) 0

b) $\frac{1}{2}$

c) 1

d) 2

e) none of the above

13. Two boats leave the same port at the same time with one boat traveling north at 15 miles per hour and the other boat traveling west at 20 miles per hour. How fast is the distance between the boats changing after 2 hours?

a) 25 miles/hr

b) 35 miles/hr

c) 5 miles/hr

d) 10 miles/hr

e) none of the above

14. Find the area of the region bounded by the curves $f(x) = x + 1$ and $g(x) = x^2 - 2x + 1$.

a) $\frac{3}{4}$

b) 9

c) $\frac{9}{2}$

d) $\frac{5}{2}$

e) none of the above

15. Find $\lim_{x\to\infty}\left(\frac{2x-1}{2x}\right)^x$.

a) $\sqrt{e}$
b) $-\sqrt{e}$
c) $\frac{1}{\sqrt{e}}$
d) $-\frac{1}{\sqrt{e}}$
e) none of the above

16. Let $h(x)=\frac{f(x)}{g(x)}$ where $f'(x)=3x^2$, $g'(x)=-2x$, $f(2)=10$, and $g(-2)=-4$. Find $h'(x)$.

a) $\frac{-x^3-4}{x^3}$
b) $\frac{-x^3+4}{x^3}$
c) $\frac{-3x+6}{x^3}$
d) $\frac{-3x-6}{x^3}$
e) none of the above

17. The price of a gallon of gasoline is currently \$3.50. A mathematical model is developed that predicts t months from now, the price will be changing at a rate of $0.005+0.015\sqrt{t}$ dollars per month. Assume this model is correct, how much will a gallon of gasoline cost four months from now?

a) \$3.60
b) \$3.65
c) \$3.70
d) \$3.75
e) none of the above

18. Suppose that f is continuous on $[a,b]$, twice differentiable in (a,b), and $f'(x)$ is never zero at any point of (a,b). Which of the following is then true?

a) f has no maximum value on $[a,b]$

b) f must have the maximum value at an interior point of the interval (a,b)

c) f has the maximum value at an endpoint , either a or b

d) f must have the maximum value at $x=a$

e) none of the above

19. Find an equation in polar coordinates for the curve $x=e^{2t}\cos t$, $y=e^{2t}\sin t$, $-\infty<t<\infty$.

a) $r=e^{\theta}$

b) $r=e^{2\sin\theta}$

c) $r=e^{2\theta}$

d) $r=e^{2\cos\theta}$

e) none of the above

20. Let $f(x)=\sin 2x$. Find all the values of x in the interval $(0,2\pi)$ such that $f(x)+f''(x)=0$.

a) $x=\dfrac{\pi}{2},\pi,\dfrac{3\pi}{2}$

b) $x=\dfrac{\pi}{4},\dfrac{5\pi}{4}$

c) $x=\dfrac{3\pi}{4},\dfrac{7\pi}{4}$

d) $x=\dfrac{\pi}{4},\dfrac{3\pi}{4},\dfrac{5\pi}{4},\dfrac{7\pi}{4}$

e) none of the above

21. Find $\int \frac{1}{\sin^2 x + 4\cos^2 x}\, dx$.

a) $\frac{1}{2}\tan^{-1}\left(\frac{\sin^2 x}{2}\right) + C$

b) $\frac{1}{2}\tan^{-1}\left(\frac{4\cos^2 x}{2}\right) + C$

c) $\frac{1}{2}\tan^{-1}\left(\frac{\tan x}{2}\right) + C$

d) $\frac{1}{2}\tan^{-1}\left(\frac{\sec x}{2}\right) + C$

e) none of the above

22. A manufacturer has determined that the total cost C of operating a factory is given by $C(x) = 0.5x^2 - 15x + 5000$ where x is the number of units produced. At what level of production will the <u>average</u> cost per unit be minimized?

a) 100

b) 95

c) 110

d) 125

e) none of the above

23. If $x^2 = 1 + \int_1^x \sqrt{1 + [f(t)]^2}\, dt$ for all $x > 1$, then find $[f(x)]^2$.

a) $4x^2 - 1$

b) $\sqrt{4x^2 - 1}$

c) $x^2 - 1$

d) There is no function f that satisfies this condition.

e) none of the above

24. Let f be continuous on $[a,b]$ and twice differentiable on (a,b). If there exists a number c such that $a<c<b$, and $f'(c)=0$, which of the following must be true?

a) $f(a)<f(b)$

b) $f(a)=f(b)$

c) $f(a)>f(b)$

d) $f(c)=\dfrac{f(b)-f(a)}{2}$

e) none of the above

25. Find the limit: $\lim\limits_{x\to 5}\dfrac{\left(\sqrt{2x-1}-3\right)}{x-5}$.

a) $\dfrac{1}{2}$

b) $\dfrac{1}{3}$

c) $\dfrac{1}{5}$

d) does not exist

e) none of the above

26. Consider the function $f(x)=x^2-3x+|3x-9|$.

a) This function is differentiable for all real numbers x.

b) This function is differentiable for all real numbers x except $x=0$.

c) This function is differentiable for all real numbers x except $x=3$ and $x=-3$.

d) This function is differentiable for all real numbers x except $x=3$.

e) none of the above

27. Find $\int \frac{e^x}{1+e^{2x}}\, dx$.

a) $\arctan\left(e^{2x}\right)+C$

b) $\arctan\left(e^{x}\right)+C$

c) $\frac{1}{2}\ln\left|1+e^{2x}\right|+C$

d) $\frac{1}{2}\ln\left|1+e^{x}\right|+C$

e) none of the above

28. A right triangle in the first quadrant has the coordinate axes as sides, and the hypotenuse passes through the point $(1, 8)$. Find the vertices of the triangle such that the length of the hypotenuse is minimized.

a) $(0,5),(10,0),(0,0)$

b) $(5,0),(0,10),(0,0)$

c) $(0,0),(0,9.6),(6,0)$

d) $(0,0),(0,11),(3,0)$

e) none of the above

29. Find a value of c such that the line $2x+y=c$ is tangent to the parabola given by the equation $y=4x^2+3$.

a) $\frac{11}{4}$

b) $-\frac{15}{4}$

c) $\frac{9}{4}$

d) $\frac{15}{4}$

e) none of the above

30. Find $\frac{dF}{dx}$ when $F(x) = \int_0^{\sin x} \sqrt{t}\, dt$.

a) $(-\sin x)\sqrt{\cos x}$
b) $(\sin x)\sqrt{\cos x}$
c) $(-\cos x)\sqrt{\sin x}$
d) $(\cos x)\sqrt{\sin x}$
e) none of the above

31. Gabriel's Horn is the name given to the solid formed by revolving the unbounded region under the curve $y = \frac{1}{x}$ for $x \geq 1$ about the x-axis. Find the volume of Gabriel's Horn.

a) π
b) 2π
c) 3π
d) ∞
e) none of the above

32. Use differentials to approximate $\sqrt{16.5}$.

a) 4.0630
b) 4.0625
c) 4.0620
d) 4.1250
e) none of the above

33. Let $f(x) = \begin{cases} 1 & \text{if } x \text{ is not an integer} \\ 1 + (-1)^x & \text{if } x \text{ is an integer} \end{cases}$

What is the left-hand limit of this function if x approaches any integer n?

a) The limit does not exist.
b) 2
c) 1
d) 0
e) none of the above

34. Find $\int \frac{\sec^3\theta \tan\theta}{1+\tan^2\theta}\,d\theta$.

a) $\frac{1}{4}\sec^4\theta + C$

b) $\frac{1}{2}\sec^2\theta + C$

c) $\frac{1}{4}\sec^2\theta\tan^2\theta + C$

d) $\sec\theta + C$

e) none of the above

35. Two towns, A and B, are on opposite sides of a river with constant width **w**. As shown in the figure below, town A is 4 miles from the river, town B is 0 miles from the river, and B is 10 miles down the river from A. Determine where a bridge (perpendicular to the river's banks) should be built over the river so that the distance between towns A and B is as short as possible. That is, find x (in miles) in the figure.

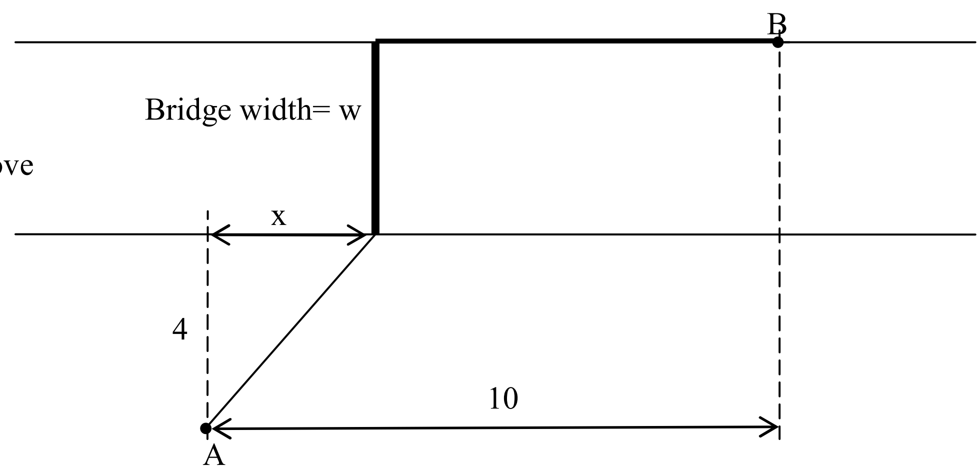
a) 1.67 *miles*

b) 10 *miles*

c) 1.23 *miles*

d) 4 *miles*

e) none of the above



36. Suppose $f(x)$ is a differentiable function with $f(1)=2$, $f(2)=-2$, $f(5)=1$, $f'(1)=3$, and $f'(2)=5$. Which of the following must be an equation of the tangent line to the graph of *f*?

a) $y-3=2(x-1)$

b) $y-2=(x-1)$

c) $y-3=5(x-1)$

d) $y-2=3(x-1)$

e) none of the above

37. Ellipse E is centered at the origin and has a horizontal minor axis of length 4. If you rotate the portion of E which falls only in the first and second quadrants about the x-axis, the resulting rotational solid has volume $\frac{800\pi}{27}$. Find the length of E's major axis.

a) $\frac{20}{3}$
b) $\frac{10}{3}$
c) 4
d) 8
e) none of the above

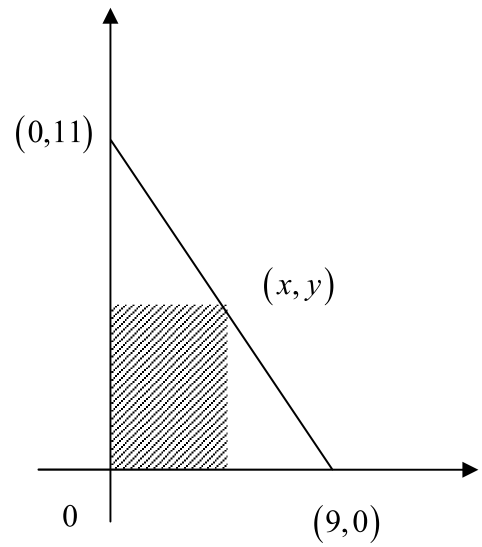
38. Let f be a twice continuously differentiable function that is concave up on the interval $[3, 5]$ and concave down on the interval $[-1, 3]$. Which of the following statements is TRUE?

a) $f''(2) > 0$ and $f''(4) < 0$
b) $f''(2) < 0$ and $f''(4) > 0$
c) $f''(3) > 0$ and $(3, f(3))$ is a point of inflection
d) Both a) and c) are true.
e) none of the above

39. Find the derivative of $f(x) = x^{(x^x)}$.

a) $x^x x^{(x^x - 1)}$
b) $x^{(x^x - 1)} x^x \left[\frac{1}{x} + 1 + \ln x\right]$
c) $x^{(x^x)} x^x \left[\frac{1}{x} + (1 + \ln x)\ln x\right]$
d) $x^x x^{(x^x - 1)} \ln x$
e) none of the above

40. A rectangle has one vertex at $(0, 0)$ and the opposite vertex lies in the first quadrant on the line passing through $(0, 11)$ and $(9, 0)$. Find the area of the largest such rectangle.



a) $\frac{33}{2}$

b) $\frac{99}{4}$

c) $\frac{99}{2}$

d) $\frac{33}{4}$

e) none of the above