Gainesville State College

Fifteenth Annual Mathematics Tournament

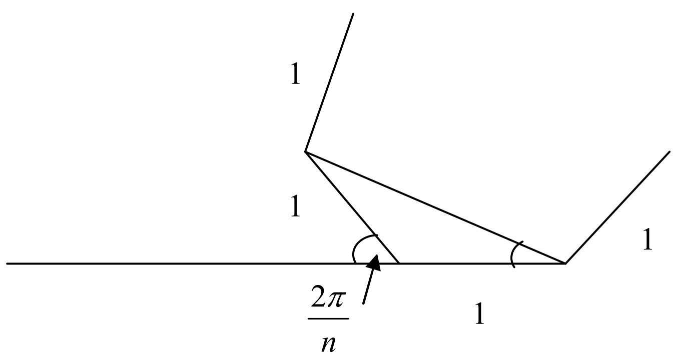
April 4, 2009

Solutions for the Afternoon Team Competition

Round 1

There are 6 terms that repeat in the sequence (5, 0, 0, 0, -5, -5). When you divide 3997 by 6, the quotient is 666 with a remainder of 1. So the 1st term in the sequence is 5.

Round 2



For a regular n-sided polygon, the external angle is $\frac{2\pi}{n}$.

The shortest diagonal will be (as seen in figure) obtained if a triangle is formed.

Thus, the shortest diagonal is the base of an isosceles triangle with external angle $\frac{2\pi}{n}$.

The base angles must be $\frac{1}{2}\left(\frac{2\pi}{n}\right)=\frac{\pi}{n}$.

Therefore, the base is $2\cos\frac{\pi}{n}$.

Round 3

On each side there are 19 x 19 small cubes with exactly one side painted, so the total of these is $6\,x\,19\,x\,19 \;=\; 2{,}166$.

Then, on each edge we have 19 cubes that have two sides painted for a total of $12\,x\,19 \;=\; 228$ cubes with two sides painted.

Finally, on each corner we have a cube with three sides painted for a total of 8 of these.

The total is $2{,}166+228+8 \;=\; 2{,}402$.

Round 4

$\theta = \dfrac{6}{16}$ radians

$$A \;=\; \frac{1}{2}r^2\theta \;=\; \frac{1}{2}(16)^2\left(\frac{6}{16}\right) = \frac{1}{2}(16)(6) \;=\; 16\cdot 3 \;=\; 48$$

Round 5

It will take the runner $\frac{1}{5}$ of an hour to complete his run along the train. His speed relative to the Earth is 25 *mph*, so the distance will be $D \;=\; \frac{1}{5}\,hr * 25\,mph \;=\; 5\,miles$

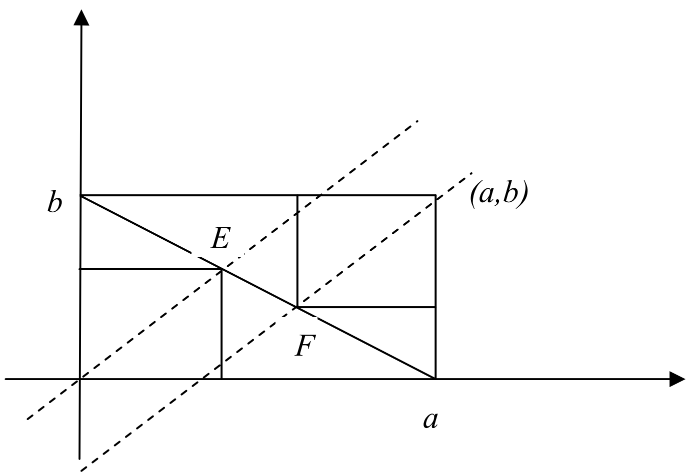
Round 6

Let the length of a side of the original cube be a. The increased side will be $1.02a$, and the new volume $(1.02)^3 a^3$. Since $(1.02)^3 = 1.061208$, the volume of the new cube is larger by approximately 6.12%.

Round 7

The big circle has an area of 4π. Each of the smaller circles has an area of π. So the two circles are each one-fourth of the larger circle. Due to the symmetry of the shape, the two "tails" must also have area equal to π. So the ratio of the area of the white circle to the shaded "tail" is $\dfrac{\pi}{\pi} = 1$.

Round 8



From the diagram, vertex E must lie on the line $y = x$ and $y = -\frac{b}{a}x + b$ (as a point on $y = -\frac{b}{a}x + b$). So it has coordinates $\left(\frac{ab}{a+b}, \frac{ab}{a+b}\right)$ found by solving these equations simultaneously.

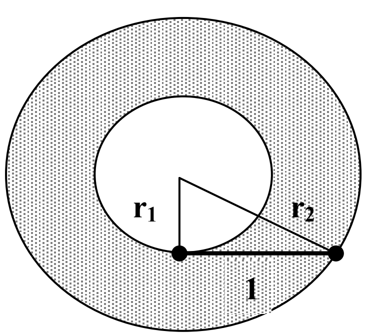
By symmetry, we must have coordinates for vertex F as $\left(\frac{a^2}{a+b}, \frac{b^2}{a+b}\right)$.

Applying the distance formula, we get

$$d\left(EF\right) = \frac{\sqrt{a^2\left(a-b\right)^2 + b^2\left(b-a\right)^2}}{a+b} = \frac{\sqrt{\left(a-b\right)^2\left(a^2+b^2\right)}}{a+b} = \frac{\left|a-b\right|\sqrt{a^2+b^2}}{a+b}, a > b$$

$$= \frac{\left(a-b\right)\sqrt{a^2+b^2}}{a+b}$$

Round 9

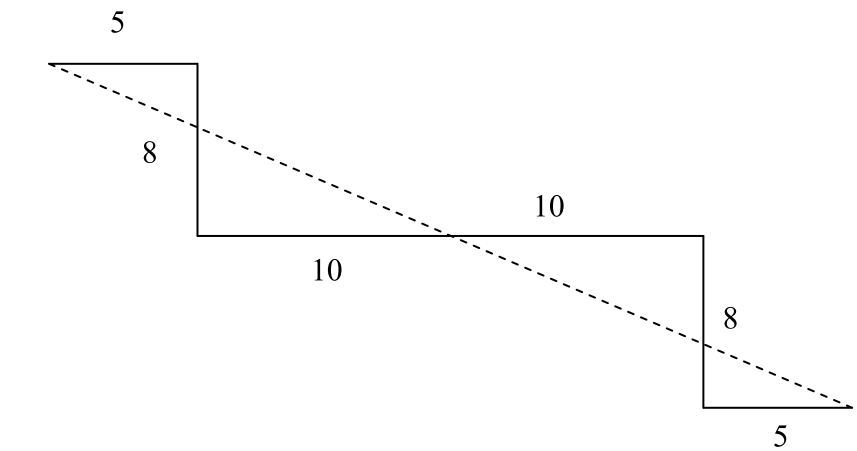


The area of the annulus is the difference of the areas of the circles, so:

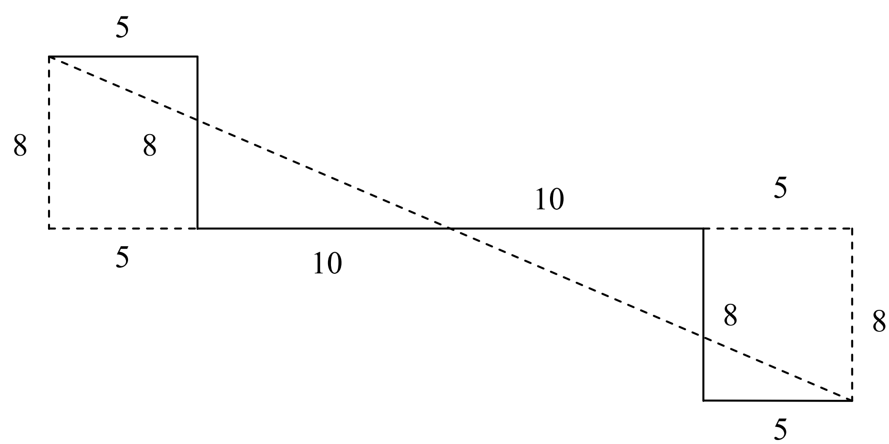
$$A = \pi r_2^{\,2} - \pi r_1^{\,2} = \pi\left(r_2^{\,2} - r_1^{\,2}\right) = \pi\left(\left[\sqrt{r_1^{\,2}+1}\right]^2 - r_1^{\,2}\right) = \pi\left(r_1^{\,2} + 1 - r_1^{\,2}\right) = \pi$$

Round 10

Here's a picture of their walks:



Notice that their destinations could have been arrived at by this dashed path instead:



Now use the Pythagorean Theorem:

$15^2 + 8^2 = 17^2$

So each hypotenuse is 17 feet ling so they are 34 feet apart.