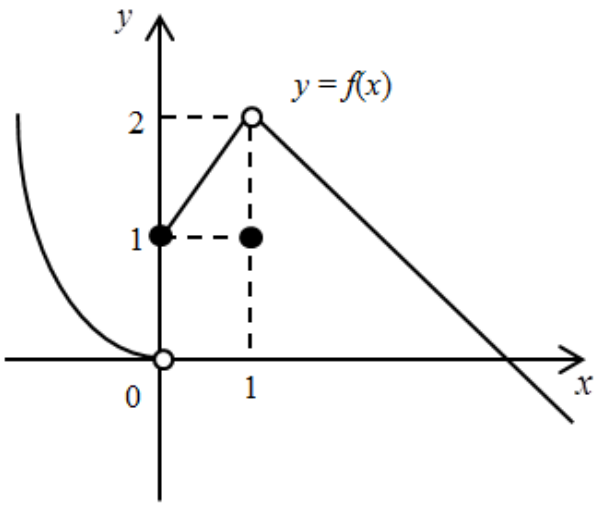
Eighteenth Annual Gainesville State College Mathematics Tournament

You may write in this test booklet. Only the electronic form will be graded. Correct answers are awarded one point. Incorrect or blank answers are awarded 0 points.

1. The following is the graph of $y = f(x)$.



Which of the following are TRUE?

I $\lim_{x \to 0^+} f(x) = 1$

II $\lim_{x \to 1} f(x) = f(1)$

III The function $g(x) = (x-1) f(x)$ is continuous at $x = 1$.

a) I

b) I, II

c) I, III

d) I, II, III

e) None of the above

2. Suppose $f''(x) = 2$ for all x on the interval $[-2, 2]$. Find the value of x in $[-2, 2]$ at which the Mean Value Theorem is satisfied.

   a) $x = 0$

   b) $x = 1$

   c) $x = \sqrt{2}$

   d) There may be more than one value of x in $[-2, 2]$ at which the Mean Value Theorem is satisfied.

   e) None of the above

3. Let $f(x) = x^x$ for $x > 0$. Find x for which $f(x) = f'(x)$.

   a) 0

   b) 1

   c) 2

   d) 3

   e) None of the above

4. Which of the following functions has a vertical tangent line at $x = 0$?

   a) $f(x) = \dfrac{1}{x}$

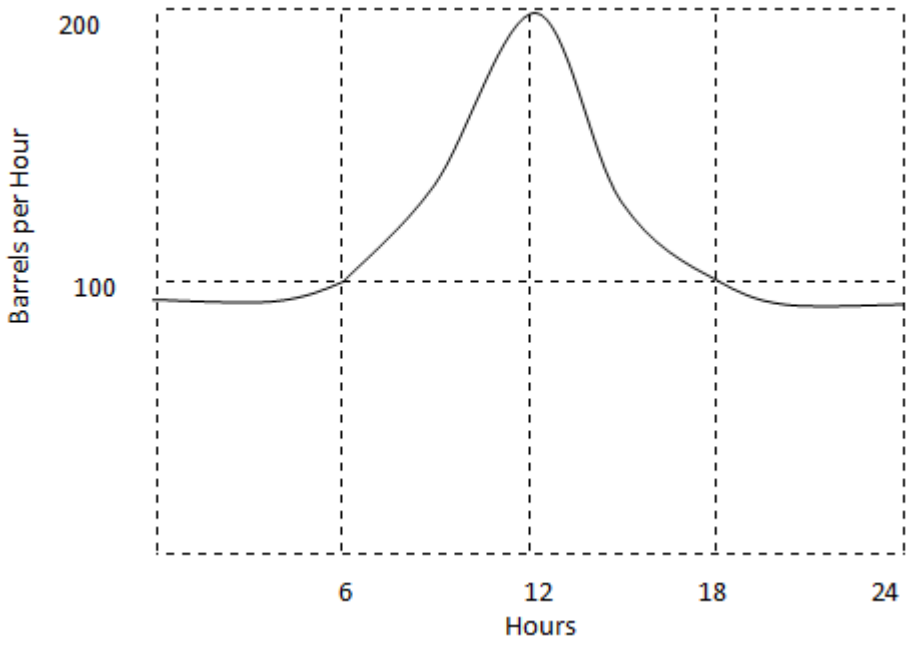
   b) $g(x) = x^{2/3}$

   c) $h(x) = x^{3/5}$

   d) All of the above

   e) None of the above

5. The flow of oil (in barrels per hour) through a pipeline on April 23rd is given by the graph below. Of the following, which best approximates the total number of barrels of oil that passed through the pipeline that day?



a) 500
b) 2400
c) 3000
d) 4800
e) None of the above

6. Define $f(1)$ in a way that extends $f(x)=\dfrac{x^3-1}{1-x^2}$ to be continuous at $x=1$.

a) $-\dfrac{3}{2}$
b) $-\dfrac{1}{2}$
c) $\dfrac{1}{2}$
d) $\dfrac{3}{2}$
e) None of the above

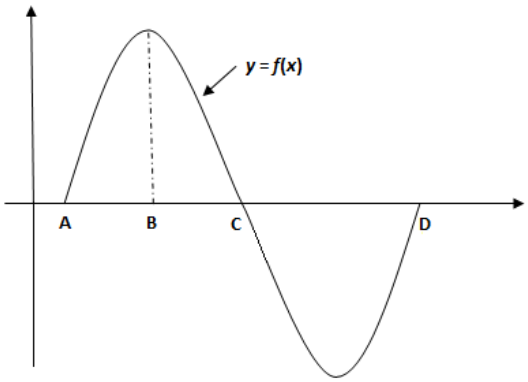
7. Suppose f is a quadratic function for which $f(0) = -1$ and $\int_{-1}^{1} f(x)\,dx = \int_{0}^{1} f(x)\,dx = \int_{-1}^{0} f(x)\,dx$. Find $f(2)$.

a) 11
b) 10
c) 9
d) 8
e) None of the above

8. Find the speed v (in miles per hour) that will minimize delivery costs on a 110-mile trip, if the cost (in dollars per hour) for fuel for the van is $C = \frac{v^2}{600}$ and the driver is paid 5 dollars per hour. (Assume there are no costs other than wages and fuel.)

a) 65.8 mi per hr
b) 55.8 mi per hr
c) 50.8 mi per hr
d) 54.8 mi per hr
e) None of the above

9. The graph of $y = f(x)$ is shown in the figure. If $g(x) = \int_{A}^{x} f(t)\,dt,$ for what value of x does $g(x)$ attain its maximum?



a) A
b) B
c) C
d) D
e) None of the above

10. Find the volume of the largest right circular cone that can be inscribed in a sphere of radius $r = \sqrt[3]{81}$.

a) 32π

b) 20π

c) $\dfrac{\pi}{81}$

d) 81

e) None of the above

11. Find all critical numbers of the greatest integer function, $f(x) = \lfloor x \rfloor$.

a) All integers
b) All real numbers that are not integers
c) All real numbers
d) Critical numbers cannot be determined.
e) None of the above

12. If f and g are twice differentiable functions and if $h(x) = f\left(g(x)\right)$, then $h''(x) =$

a) $f''\left(g(x)\right)\left[g'(x)\right]^2 + f'\left(g(x)\right)g''(x)$

b) $f''\left(g(x)\right)g'(x) + f'\left(g(x)\right)g''(x)$

c) $f''\left(g(x)\right)\left[g'(x)\right]^2$

d) $f''\left(g(x)\right)g''(x)$

e) None of the above

13. Let $f(x) = \left| b - x^2 \right|$ for a constant $b > 0$. Is f continuous and/or differentiable at $x = \sqrt{b}$?

a) f is continuous and differentiable at $x = \sqrt{b}$.

b) f is neither continuous nor differentiable at $x = \sqrt{b}$.

c) f is continuous but not differentiable at $x = \sqrt{b}$.

d) f is differentiable but not continuous at $x = \sqrt{b}$.

e) Not possible to determine from the information given

14. Find the limit: $\lim_{x\to\frac{\pi}{2}^+}\left[\ln\left(\sqrt{\tan^2(x)+1}+\tan(x)\right)-\ln\left(\tan(x)\right)\right]$.

a) 0

b) $\ln 2$

c) $\ln(1/2)$

d) ∞

e) None of the above

15. The base of the solid is a region in the first quadrant bounded by the x-axis, the y- axis, and the line $x+2y=8$. If the cross sections of the solid perpendicular to the x-axis are semicircles, what is the volume of the solid?

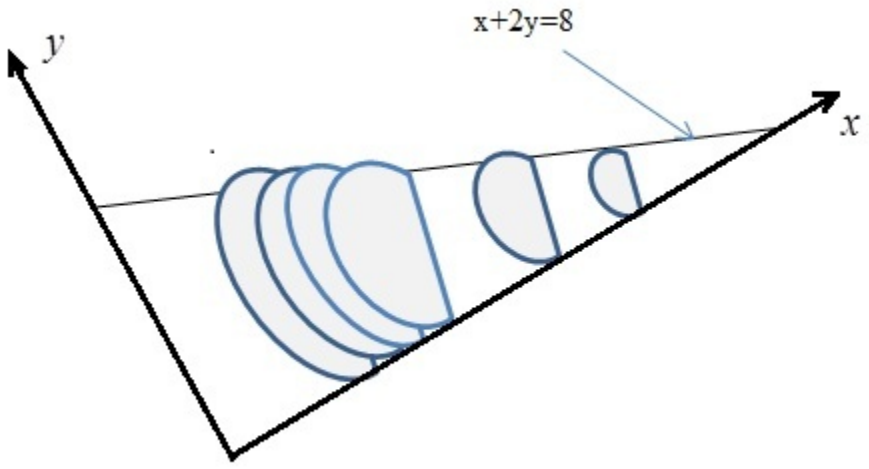
a) 12.66

b) 14.661

c) 16.755

d) 67.021

e) None of the above



16. Evaluate $\int_1^2 \frac{dx}{\sqrt{x(x-1)}}$.

a) $2\ln\left(\sqrt{2}-1\right)$

b) $2\ln\left(\sqrt{2}+1\right)$

c) $\frac{1}{\sqrt{2}}\ln\left(\sqrt{2}+1\right)$

d) $\frac{1}{\sqrt{2}}\ln\left(\sqrt{2}-1\right)$

e) None of the above

17. Suppose f and g are functions and $f(3)=2,\ f'(3)=4,\ g(5)=3,\ g'(5)=7.$

Find $\left(f\circ g\right)'(5)$.

a) 12
b) 14
c) 21
d) 28
e) None of the above

18. Evaluate $\int_{-2011}^{2011} (x^{2011}e^{-\frac{x^2}{2}} + \sin^{2011}x - 2011x)\,dx$.

a) 0
b) -1
c) 1
d) -2
e) None of the above

19. Find the minimum value of the function $f(x)=e^x - x - \frac{x^3}{3}$.

a) 0
b) -1
c) $\frac{4}{3}$
d) 1
e) None of the above

20. If $f(x)=(x-2)^4(x-3)^3(x-4)^2$, find $f'''(2)+f''(3)+f'(4)$.

a) 16

b) 27

c) 0

d) 16

e) None of the above

21. Let $3-2\sqrt{x}=\int_0^{\sqrt{x}} f(t)\,dt$. Find $f(2)$.

a) $2\sqrt{2}$

b) $3-2\sqrt{2}$

c) 3

d) -2

e) None of the above

22. Find the limit: $\lim\limits_{x\to\frac{\pi}{2}^-} \dfrac{\frac{\pi}{2}-x}{\sin\left(\frac{\pi}{2}-x\right)}$.

a) DNE

b) 0

c) -1

d) 1

e) None of the above

23. Compute the area between the x-axis and the graph of $f(x)=\left|e^x-1\right|$ on the interval $[-1,2]$.

a) 2

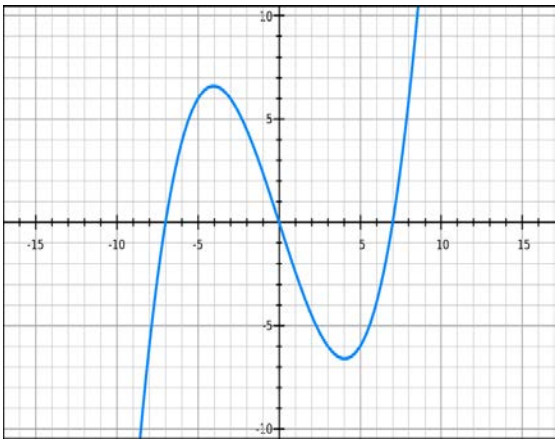
b) $e^2+e^{-1}-3$

c) e^2-1

d) ∞

e) None of the above

24. Consider the following graph of a function f that is differentiable for all real numbers.



Define a new function $H(x) = 2f(3-x)$. What can be said about $H'(6)$?

a) $H'(6) > 0$

b) $H'(6) < 0$

c) $H'(6) = 0$

d) $H'(6)$ DNE

e) None of the above

Reminder

Question 25 will be used as a tie-breaker, if necessary.

25. Find the limit: $\lim_{n\to\infty}\left(\frac{1}{n+1}+\frac{1}{n+3}+\cdots+\frac{1}{n+(2n-1)}\right)$.

a) $\frac{1}{2}\ln 2$

b) $\ln 2$

c) $\frac{1}{2}\ln 3$

d) $\ln 3$

e) None of the above

26. Find the arc length of the graph of $y = \frac{x^3}{6} + \frac{1}{2x}$ on the interval $\left[\frac{1}{2}, 2\right]$.

a) $\frac{33}{16}$

b) $\frac{35}{16}$

c) $\frac{31}{8}$

d) $\frac{31}{24}$

e) None of the above

27. Evaluate $\int_0^{\frac{\pi}{2}} x\sin(x)\cos(x)\,dx$.

a) $\frac{\pi}{2}$

b) 0

c) $\frac{\pi}{8}$

d) 1

e) None of the above

28. Consider the function $f(x) = \frac{x^5 e^x (4x+3)}{5^{\ln x}(3-x)^2}$. Which of the following is an equation of the line tangent to the graph of f at $x = 1$?

a) $y - (7/4)e = (7/4)e\ (51/14 + \ln 5)(x-1)$

b) $y - (7/4)e = (7/4)e\ (55/7 - \ln 5)(x-1)$

c) $y - (7/4)e = (7/4)e\ (57/14 + \ln 5)(x-1)$

d) $y - (7/4)e = (7/4)e\ (53/7 - \ln 5)(x-1)$

e) None of the above

29. Along the graph of the equation $y = x^3 - 6x^2 + 3x + 5$, both the coordinate y and the slope m change, but generally at different rates. Find the coordinate x of the point or points, if any, where the coordinate y and the slope are momentarily changing at the same rate.

a) $x = 5,\ x = 1$

b) $x = 3,\ x = -6$

c) $x = \frac{4}{3}$

d) $x = 2 + \sqrt{3},\ x = 2 - \sqrt{3}$

e) None of the above

30. Find the limit: $\lim_{x \to 0^+} (\sin x)^{\tan x}$.

a) 0

b) 1

c) $\frac{1}{2}$

d) $\frac{1}{3}$

e) None of the above

31. Find $\int \frac{1}{x^2 - 6x + 8}\, dx$.

a) $\frac{1}{2} \ln \left| \frac{x-4}{x-2} \right| + C$

b) $\frac{1}{2} \ln \left| \frac{x-2}{x-4} \right| + C$

c) $\frac{1}{2} \ln \left| (x-2)(x-4) \right| + C$

d) $\ln \left| (x-2)(x-4) \right| + C$

e) None of the above

32. Let $f(x) = \int_0^{x^2} \sin t \, dt$. At how many points in the closed interval $\left[0, \sqrt{\pi}\right]$ does the instantaneous rate of change of f equal the average rate of change of f on the interval?

a) Zero
b) One
c) Two
d) Three
e) None of the above

33. Let $f(x) = x^3 - 3x^2 - 1, \quad x \geq 2$. Find $\left(f^{-1}\right)'(-1)$.

a) -1
b) 2
c) $\frac{1}{2}$
d) $\frac{1}{9}$
e) None of the above

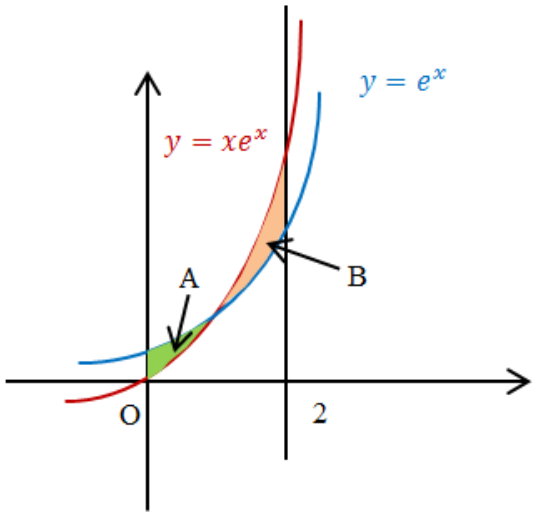
34. Evaluate $\int_0^{\pi/2} \frac{\sin^3 x}{\sin^3 x + \cos^3 x} dx$.

a) 0
b) $\pi/4$
c) $\pi/2$
d) 1
e) None of the above

35. Find the limit: $\lim_{h \to 0} \left(\frac{\int_1^{1+h} \sqrt{x^5 + 8} \, dx}{h} \right)$.

a) 3
b) $2\sqrt{2}$
c) 1
d) 0
e) None of the above

36. The areas A and B are bounded by the graphs of $y = e^x$, $y = xe^x$, $x = 0$, and $x = 2$, as in the picture.



Find the value of $B - A$.

a) e^{-1}
b) 2
c) e
d) $\frac{5}{2}$
e) None of the above

37. Find $\int \frac{dx}{x^{2/3} - x^{1/2}}$.

a) $2x^{1/2} - 3x^{1/3} - 6\ln\left|x^{1/6} - 1\right| + C$
b) $3x^{1/3} - 2x^{1/2} - 6\ln\left|x^{1/6} - 1\right| + C$
c) $2x^{1/2} + 3x^{1/3} + 6\ln\left|x^{1/6} - 1\right| + C$
d) $3x^{1/3} + 6x^{1/6} + 6\ln\left|x^{1/6} - 1\right| + C$
e) None of the above

38. Evaluate $\int_0^{(\sqrt{2}-1)/2} \frac{1}{(2x+1)\sqrt{x^2+x}}\,dx$.

a) 0

b) $\frac{\pi}{4}$

c) $\frac{\pi}{3}$

d) $\frac{\pi}{2}$

e) None of the above

39. Consider a 30-foot chain that weighs 4 pounds per foot hanging from a winch 30 feet above ground level. Find the work done by the winch in winding up the entire chain with a 700-pound load attached to it.

a) $21,000$ ft-lb

b) $21,120$ ft-lb

c) 1800 ft-lb

d) $22,800$ ft-lb

e) None of the above

40. Evaluate $\int_{-\infty}^{\infty} \frac{1}{x^2-6x+10}\,dx$.

a) $\frac{\pi}{4}$

b) $\frac{\pi}{2}$

c) π

d) 2π

e) None of the above