University of North Georgia

Sophomore Level Mathematics Tournament

April 5, 2014

Solutions for the Afternoon Team Competition

Round 1

Volume = $\pi r^2 h = \pi \cdot 6^2 \cdot 3 = \pi \cdot 6 \cdot 6 \cdot 3 = \pi \cdot 6 \cdot 2 \cdot 3 \cdot 3 = 12 \cdot 9\pi$
The answer is 12 pieces.

Round 2

We think about the complement – people choose different numbers.
The first person can choose any number (positive integer less than 11: from 1 to 10), then the second person would have 9 (different) numbers to choose (9/10), the third person 8 (different) numbers to choose, etc. So the probability that the 4 people choose different numbers is:
$1 \cdot \frac{9}{10} \cdot \frac{8}{10} \cdot \frac{7}{10} = \frac{504}{1000}$. Hence the probability that two of the people choose the same number is:
$1 - \frac{504}{1000} = \frac{496}{1000} = 0.496$.

Round 3

Since $f(x)$ is divisible by $(x-1)^3$, $x^4 + ax^2 + bx + c = (x-1)^3(x-d)$ for some real number *d*.
Now if we equate the coefficient of x^3 on both sides we see that $d = -3$.
Then $f(2) = (2-1)^3(2-(-3)) = 5$.

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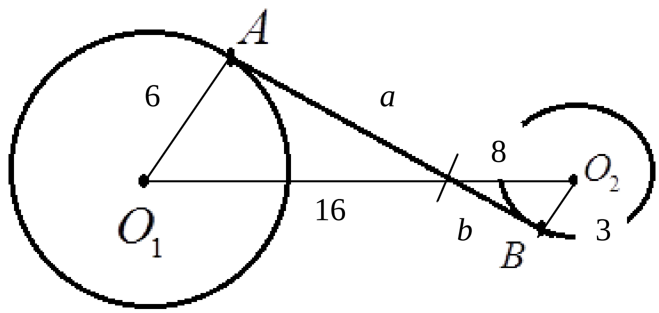
Round 4



We get 16 and 8 from the fact that the triangles are congruent. Then we use the Pythagorean Theorem twice getting $a=\sqrt{220}=2\sqrt{55}$ and $b=\sqrt{55}$. So $a+b=3\sqrt{55}$.

Round 5

We have $\cot\alpha+\cot\beta=4$, so $\dfrac{1}{\tan\alpha}+\dfrac{1}{\tan\beta}=4$ and $\dfrac{\tan\beta+\tan\alpha}{\tan\alpha\tan\beta}=4$.

Thus, $\tan\alpha\tan\beta=\dfrac{\tan\alpha+\tan\beta}{4}=\dfrac{7}{4}$.

Then $\tan(\alpha+\beta)=\dfrac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}=\dfrac{7}{1-\frac{7}{4}}=\dfrac{28}{4-7}=-\dfrac{28}{3}$

Round 6

Let *r* be the radius in inches. Then the area in square inches is πr^2 which must be a natural number according to the problem.

Since $2.54=\dfrac{127}{50}$, the area in square centimeters is $\pi\left(\dfrac{127}{50}r\right)^2=\dfrac{16129}{2500}\pi r^2$. The smallest natural number value of πr^2 for which this is an integer is 2500. So $\pi r^2=2500$ and $r^2=\dfrac{2500}{\pi}$.

Then $r=\dfrac{50}{\sqrt{\pi}}$ inches.

Round 7

$f(1) = 2 + 4 + 6 + \cdots + 100 = (2 + 4 + 6 + \cdots + 98) + 100$ and

$g(1) = 1 + 3 + 5 + \cdots + 99 = 1 + (2 + 1 + 4 + 1 \cdots + 98 + 1) = 50 + (2 + 4 + 6 + \cdots + 98)$

The sum $(2 + 4 + 6 + \cdots + 98)$ can be evaluated as $2(1 + 2 + 3 + \cdots + 49) = 49(50) = 2450$.

Consequently, $f(1) = 2450 + 100 = 2550$ and $g(1) = 50 + 2450 = 2500$.

So $f^2(1) - g^2(1) = (f(1) + g(1))(f(1) - g(1)) = (2550 + 2500)(2550 - 2500) = 252{,}500$

Dividing 252,500 by 100 gives 2525.

Round 8

We are looking for $abcd < 1200$, where *a*, *b*, *c*, and *d* are primes with $a < b < c < d$. We solve this problem by finding the largest possible value for *a*, then for *b*, and so on. It turns out you can find the answer by making a dozen or so calculations.

1. Establish a benchmark by multiplying consecutive primes:

$$2 \cdot 3 \cdot 5 \cdot 7 = 210$$
$$3 \cdot 5 \cdot 7 \cdot 11 = 1155$$
$$5 \cdot 7 \cdot 11 \cdot 13 = 5005$$

which is the smallest value of *abcd* where $a > 3$, but it's too big. So a is 2 or 3, and our current benchmark for *abcd* is 1155.

2. $3 \cdot 5 \cdot 7 \cdot 13$ is the smallest number involving $a = 3$ that we haven't checked yet, but it's 1365 which is too big. So the only remaining numbers to check have $a = 2$, which means $bcd < 600$ where *b* is at least 3.

From now on we are assuming $a = 2$ and we want to find the largest value of $bcd < 600$.

3. Establish a benchmark for *bcd* by multiplying consecutive primes:

$$3 \cdot 5 \cdot 7 = 105$$
$$5 \cdot 7 \cdot 11 = 385$$
$$7 \cdot 11 \cdot 13 = 1001$$

which is the smallest value of *bcd* where $b > 5$, but it's too big. So *b* is 3 or 5.

4. Assuming $b = 5$:

$$5 \cdot 7 \cdot 13 = 455$$
$$5 \cdot 7 \cdot 17 = 595$$

which is the largest number less than 600 divisible by 5. This is our new benchmark for *bcd*.

5. The only number greater than 595 but less than 600 that is divisible by 3 is 597, which isn't a product of three primes. So we don't need to check the possibility of $b = 3$.

Thus $2 \cdot 5 \cdot 7 \cdot 17 = 1190$, which is larger than our previous *abcd* benchmark of 1155.

Round 9

Note that paths cannot be repeated. We will count all the possible paths from *S* to *F* that pass through *M* or *N* separately and then subtract any paths that are repeated. This is known as an inclusion-exclusion method.

Part 1: Paths from *S* to *F* through *M* (or simply *SMF* paths) – these go from *S* to *M* and then to *F*. There are exactly 3 paths from *S* to *M* (of length 3 each). There are exactly 10 paths from *M* to *F* (of length 5 each). For each of the 2 *SM* paths, there are 10 *MF* paths giving a total of $3 \cdot 10 = 30$ *SMF* paths.

Part 2: Paths from *S* to *F* through *N* (or simply *SNF* paths) – these go from *S* to *N* and then to *F*. There are 15 paths from *S* to *N* (of length 6 each). There are 2 paths from *N* to *F* (of length 2 each). For each of the 15 *SN* paths, there are 2 *NF* paths giving a total of $15 \cdot 2 = 30$ *SNF* paths.

Part 3: Paths through both *M* and *N* together (or simply *SMNF* paths) – these go from *S* to *M* then *M* to *N* then *N* to *F*. From part 1, we have 2 *SM* paths (each of length 3). There are only 3 paths from *M* to *N* (each of length 3). From part 2, we have 2 *NF* paths (each of length 2). For each of the 3 *SM* paths, there are 3 *MN* paths for a total of $3 \cdot 3 = 9$ *SMN* paths. For each of these 9 paths, there are 2 *NF* paths, so there will be a total of $9 \cdot 2 = 18$ *SMNF* paths.

Notice that the *MN* paths have been counted twice and so we have to take out one of them. So there are $30 + 30 - 18 = 42$ paths from *S* to *F* that pass through *M* or *N*.

This solution can also be written using Combinations:

$$C(3,2) \cdot C(5,3) + C(6,4) \cdot C(2,1) - C(3,2) \cdot C(3,2) \cdot C(2,1) = 30 + 30 - 18 = 42.$$

Round 10

For the logarithm with base between 0 and 1 to be positive, the argument must be between 0 and 1, so we get $0 < \frac{1}{x^2 - 2} < 1$. To satisfy the "left" side of the above, $x^2 - 2$ must be positive. To satisfy the "right" side, $x^2 - 2$ must be larger than 1. The condition $x^2 - 2$ larger than 1 is equivalent to both of these conditions, so we get $x^2 - 2 > 1$ which gives $x^2 > 3$.
The solution is $\left(-\infty, -\sqrt{3}\right) \cup \left(\sqrt{3}, \infty\right)$.