University of North Georgia
Twentieth Annual Sophomore Level Mathematics Tournament

You may write in this test booklet. Only the electronic form will be graded. Correct answers are awarded one point. Incorrect or blank answers are awarded 0 points.

1. Find the limit:

$$\lim_{x\to 0} \frac{\pi - \pi \cos^2 x}{x^2}.$$

a) $-\pi$

b) -2

c) π

d) 2

e) None of the above

2. The positive integers m and n do not have common factors and are chosen in such a way that

$$\lim_{x\to 0} \frac{3\sqrt{49+x} - 7\sqrt{9-x}}{x} = \frac{m}{n}.$$

What is $m+n$?

a) -41

b) 50

c) $\frac{29}{21}$

d) $-\frac{20}{21}$

e) None of the above

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3. What is the limit as $a \to 0^+$ of the larger of the two roots of the equation $ax^2 + bx + c = 0$, where a, b, c are real numbers and $b > 0$?

   a) ∞

   b) $-\infty$

   c) $-\frac{c}{b}$

   d) $4c$

   e) None of the above

4. Find the improper integral:

$$\int_1^{\infty} \frac{\sqrt{x^2 - 1}}{x^3} dx \, .$$

   a) $\sqrt{\frac{\pi}{2}}$

   b) 2π

   c) $\frac{\pi}{4}$

   d) $\frac{\pi}{2}$

   e) None of the above

5. Let $y = \frac{1}{2x}$. Find $\frac{d^n y}{dx^n}$.

   a) $\frac{1}{2^n x^n}$

   b) $\frac{n!}{2^n x^n}$, $n! = n(n-1)(n-2)(n-3) \ldots (3)(2)(1)$

   c) $\frac{(-1)^n n!}{2^n x^{n+1}}$

   d) $\frac{(-1)^n n!}{2x^{n+1}}$

   e) None of the above

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6. Find the minimum value of the function $f(x) = x^2 - 8x + 12 - 10\sqrt{x^2 - 8x + 12}$.

   a) -24
   b) -25
   c) -4
   d) 0
   e) None of the above

7. Two half lines begin at the point O, creating an angle of $60°$. From the point O, two particles start moving at the same time, each on a different half line. The first one is moving with the constant speed of 5 m/sec. The second is moving in such a way that its distance from the point O can be expressed by $s = 2t^2 + t$, where s is measured in meters and t in seconds. How fast is the distance between the two particles changing when the first particle is 10 meters from the point O?

   a) 1 m/sec
   b) 3 m/sec
   c) 5 m/sec
   d) 7 m/sec
   e) None of the above

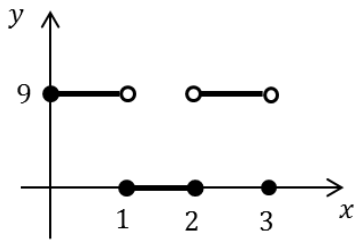
8. Given that

$$x = \int_0^y \frac{dt}{\sqrt{1 + 4t^2}}$$

   and that $\frac{d^2y}{dx^2}$ is proportional to y, determine the constant of proportionality.

   a) $\frac{1}{8}$
   b) 8
   c) $\frac{1}{4}$
   d) 4
   e) None of the above

9. Find the average value of the function represented on the graph over $[0, 3]$.



a) f is not continuous thus no average exists

b) 0

c) 4.5

d) 6

e) None of the above

10. Let $f(x) = x^{x^x}$. Find $f'(x)$.

a) $f'(x) = x^{x^x}\left(\frac{1}{x} + \ln x + (\ln x)^2\right)$

b) $f'(x) = x^{x^x}(x + 2x \ln x)$

c) $f'(x) = xx^{x^{x-1}}$

d) $f'(x) = x^{x^x} x^x\left(\frac{1}{x} + \ln x + (\ln x)^2\right)$

e) None of the above

11. Find the limit:

$$\lim_{n\to\infty} \frac{1}{n}\left(2^{\frac{1}{n}} + 2^{\frac{2}{n}} + \ldots + 2^{\frac{n}{n}}\right).$$

a) $\frac{1}{\ln 2}$

b) $2 \ln 2$

c) $\sqrt{\ln 2}$

d) $\ln 2$

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e) None of the above

12. Find all values of a that satisfy the equation

$$\int_0^a \left(x^2 - 4x + 4\right) dx = \int_a^4 \left(x^2 - 4x + 4\right) dx.$$

a) $\frac{5}{2}$

b) $\frac{11}{5}$

c) 2

d) $\frac{3}{2}$

e) None of the above

13. The graph of $y = \frac{\sin x}{x}$ has

I. a vertical asymptote at $x = 0$.

II. a horizontal asymptote at $y = 0$.

III. an infinite number of zeros.

a) Only I

b) Only II

c) Only III

d) Only II and III

e) None of the above

14. Find the limit:

$$\lim_{x \to 0} \frac{|2x - 1| - |2x + 1|}{x}.$$

a) 2

b) 4

c) -4

d) 1

e) None of the above

15. Find the derivative of the function $f(x) = (x^2 + 4x + 5) \cdot \sin x \ \ at \ \ x = 0$.

a) 4

b) 0

c) 5

d) 9

e) None of the above

16. Find the definite integral:

$$\int_0^{\frac{\pi}{4}} \frac{\sin x \cdot \cos x}{\sin^4 x + \cos^4 x} dx \, .$$

a) $\frac{\pi}{16}$

b) $\frac{\pi}{8}$

c) $\frac{\pi}{4}$

d) $\frac{\pi}{2}$

e) None of the above

17. Given that $x^2 + 4xy + 4y^2 = 1$, find $\frac{dy}{dx}$.

a) $-\frac{1}{4}$

b) $-\frac{1}{2}$

c) -2

d) -4

e) None of the above

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18. Find the improper integral:

$$\int_{1}^{\infty} (4 + 2x) \cdot e^{-x}\, dx\,.$$

a) $\frac{8}{e}$

b) $4e$

c) ∞

d) $\frac{4}{e} + 1$

e) None of the above

19. Calculate the volume of the solid generated by revolving about the *y*-axis the region bounded by $y = x^2,\ y = 0,\ x = 1$.

a) $\frac{\pi}{2}$

b) $\frac{\pi}{5}$

c) $\frac{3\pi}{2}$

d) $\frac{\pi}{3}$

e) None of the above

Reminder

Question 20 will be used as a tie-breaker, if necessary.

20. Use properties of the natural logarithm to compute

$$\int_0^1 \frac{12}{(x+3)^2} \cdot \ln\left(\frac{x+1}{x+2}\right) dx\,.$$

Leave your answers in terms of *natural logarithm* only.

a) $6\ln 3 - 2\ln 2$

b) $23\ln 2 - 15\ln 3$

c) $7\ln 2 + 6\ln 3$

d) $12\ln 3 - 17\ln 2$

e) None of the above

21. Assume f'' is continuous and $f(1) = 3,\ f'(1) = 2$ and $\int_0^1 f(x)\,dx = 5.$

Find the definite integral:

$$\int_0^1 x^2 \cdot f''(x)\,dx.$$

a) 15

b) -6

c) 10

d) 6

e) None of the above

22. Find the limit:

$$\lim_{x\to 3} \frac{f(x) - f(3)}{\frac{1}{x} - \frac{1}{3}} \quad \text{if } f'(3) = 4\,.$$

a) -4

b) 36

c) 9

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d) -9

e) None of the above

23. In L'Hopital's 1696 calculus textbook, he illustrated his rule using the limit of the function

$$f(x) = \frac{\sqrt{2a^3x - x^4} - a\sqrt[3]{a^2x}}{a - \sqrt[4]{ax^3}}$$

as x approaches a, $a > 0$. Find the limit.

a) $\frac{4a}{3}$

b) $\frac{16a}{9}$

c) 1

d) $\frac{3a}{4}$

e) None of the above

24. Given that $x = 2t^3 + 4t \quad and \quad y = 3t^2$, find $\frac{dy}{dx}$.

a) $\frac{3t}{3t^2+2}$

b) $\frac{-6t^4+12t^2}{(2t^3+4t)^2}$

c) $36t^3 + 24t$

d) 0

e) None of the above

25. Find the definite integral:

$$\int_{-\pi}^{\pi} |\sin x + \cos x| \, dx .$$

a) $\sqrt{2}$

b) $\frac{1+4\sqrt{3}}{3}$

c) $\frac{4\sqrt{3}-1}{3}$

d) $4\sqrt{2}$

e) None of the above

26. For what values of the numbers a and b does the function $f(x) = axe^{bx^2}$ have the maximum value $f(2) = 1$?

a) $a = -0.5e^{0.5}$, $b = 0.125$

b) $a = 0.5e^{-0.5}$, $b = 0.125$

c) $a = -0.5e^{-0.5}$, $b = -0.125$

d) $a = 0.5e^{0.5}$, $b = -0.125$

e) None of the above

27. Find the definite integral:

$$\int_1^e \frac{dx}{x\sqrt{\ln x}}.$$

a) 2

b) 0

c) $2\sqrt{e}$

d) $\sqrt{e}$

e) None of the above

28. The oil in a spherical tank 50 feet in diameter is 20 feet deep. How much oil does the tank contain?

a) $\frac{62500}{3}\pi$ ft^3

b) $\frac{31250}{3}\pi$ ft^3

c) $\frac{22000}{3}\pi$ ft^3

d) $\frac{78125}{24}\pi$ ft^3

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e) None of the above

29. Find the definite integral:

$$\int_1^e \frac{e^x(1+x\ln x)}{x}\,dx\,.$$

a) e^e

b) e

c) e^2

d) $\frac{1}{e}$

e) None of the above

30. Let $f(x) = x^{\frac{2}{3}} \cdot \tan x$. Which statement regarding the derivative of this function at $x = 0$ is true?

a) $f'(0)$ does not exist.

b) $f'(0) = 0$

c) $f'(0) = 1$

d) $f'(0) = \frac{\sqrt{3}}{3}$

e) None of the above

31. Find the limit:

$$\lim_{x\to\infty} \frac{e^{6x}}{\int_0^x \sqrt{t+e^{12t}}\,dt}\,.$$

a) 6

b) 12

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c) 2

d) 4

e) None of the above

32. If f is differentiable at $x = a$, which of the statements (a) to (d) could be false?

a) f is continuous at $x = a$.

b) $\lim_{x \to a} f(x)$ exists.

c) $f''(a)$ is defined.

d) $\lim_{x \to a} \frac{f(x) - f(a)}{x - a}$ exists.

e) None of the above

33. Suppose that $f(0) = 0$ and $f'(0) = 2$. Let $g(x) = f\left(-x + f\left(f(x)\right)\right)$. Find $g'(0)$.

a) 6

b) 5

c) -5

d) $\frac{3}{2}$

e) None of the above

34. Find the limit:

$$\lim_{n \to \infty} \left(\frac{n^2 + 4n + 6}{n^2 + 2n + 7} \right)^n .$$

a) e^2

b) $\ln 2$

c) 2^e

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d) $\frac{1}{e}$

e) None of the above

35. How many inflection points does the curve given by the equation $y(x^2+4) = 4(2-x)$ have?

a) No inflection points

b) Exactly one inflection point

c) Exactly two inflection points

d) Exactly three inflection points

e) None of the above

36. Suppose $f(0) = -5$ and $f'(x) \leq 7$ for any real number x. How large can $f(3)$ possible be?

a) 8

b) 16

c) 21

d) 15

e) None of the above

37. The line tangent to the graph of the function $y = x^4 + x^3 - x + 1$ at the point with the x-coordinate $x = 1$, crosses the x-axis at the point with the x-coordinate.

a) $\frac{1}{3}$

b) $\frac{1}{4}$

c) $\frac{1}{2}$

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d) $\frac{2}{3}$

e) None of the above

38. Given that $f(x) = \sqrt{x-3}$, find the definite integral:

$$\int_0^2 f^{-1}(x)\,dx\,.$$

a) $\frac{16}{3}$

b) $\frac{26}{3}$

c) $\frac{4}{3}\sqrt{2}$

d) $\frac{3}{8}\sqrt{2}$

e) None of the above

39. Find the area of the region cut off from the parabola $y = 6 + x - x^2$ by the chord joining the points $(-1, 4)$ and $(3,0)$.

a) $\frac{104}{3}$

b) $\frac{80}{3}$

c) $\frac{56}{3}$

d) $\frac{32}{3}$

e) None of the above

40. The function $f(x) = x^2e^x$ is concave down on the interval $[a, b]$, concave upward on $(-\infty, a]$, and also on $[b, \infty)$. Find $a \cdot b$.

a) 4

b) 2

c) -4

d) -2

e) None of the above