University of North Georgia

Sophomore Level Mathematics Tournament

April 11, 2015

Solutions for the Afternoon Team Competition

Round 1

If you give 1 cookie to the first friend, 2 cookies to the second friend, etc., after 19 friends you have given away $1+2+\cdots+19=190$ cookies. The 10 remaining cookies are not enough for a 20^{th} friend, so you give them to the 19^{th} friend. In fact, 20 friends would require at least $1+2+\cdots+20=210$ cookies, so the largest number of friends that can receive cookies is 19.

Round 2

Since 1 yard = 3 feet, 32 feet = $\frac{32}{3}$ yards. The area of the lawn is $A=l\cdot w=(15yd)\left(\frac{32}{3}yd\right)=160yd^2$.

So a 15% increase in the area is $0.15A=(0.15)(160yd^2)=24yd^2$.

Round 3

$P(\text{at least 1 sample is all green}) = 1-P(\text{no sample is all green})$

$= 1-P(\text{all samples} < 10 \text{ green})$

Because each sample is independent of the other,

$= 1-P(1 \text{ sample} < 10 \text{ green})^{656}$

$= 1-\left[1-P(\text{all green})\right]^{656}$

$= 1-\left[1-0.45^{10}\right]^{656}$

$= 1-\left[0.9996594937\right]^{656}$

$= 1 - 0.7997867382$

$= 0.2002132618$

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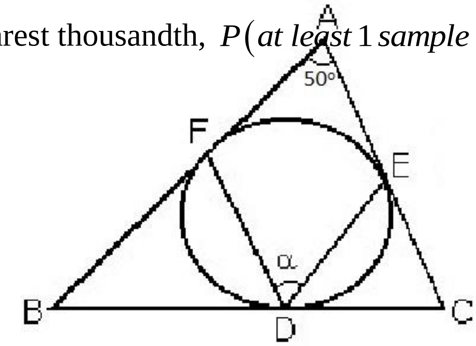
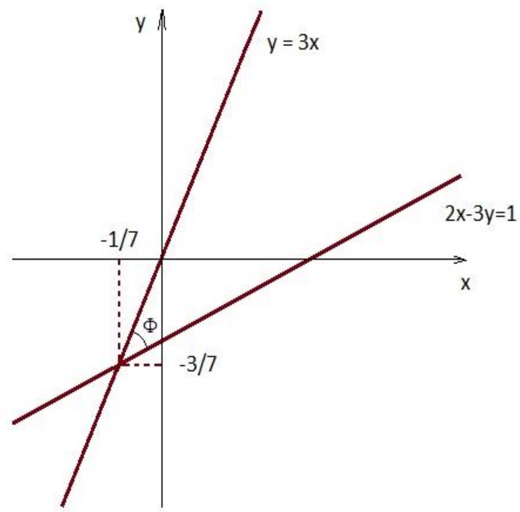
So, rounded to the nearest thousandth, $P(\text{at least 1 sample is all green}) = 0.200$.

Round 4



Let O be the center of the circle inscribed in ABC. Since $\angle FAE + \angle FOE = 180°$, then $\angle FOE = 130°$. Also, $2(\angle FDE) = \angle FOE$ implies that $\angle FDE = \alpha = 65°$.

Round 5



For the first line: $y - 3x = 0$, so $y = 3x$ and $m_1 = 3$.

For the second line: $2x - 3y = 1$, so $y = \frac{2}{3}x - \frac{1}{3}$ and $m_2 = \frac{2}{3}$.

Using the difference formula for tangent, we have

$$\tan\Phi = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right| = \left|\frac{\frac{2}{3} - 3}{1 + (3)\left(\frac{2}{3}\right)}\right| = \left|\frac{-\frac{7}{3}}{3}\right|.$$

So $\tan\Phi = \frac{7}{9}$, then $\Phi = \tan^{-1}\left(\frac{7}{9}\right) = 37.87°$.

Rounded to the nearest whole degree $\Phi = 38°$

Round 6

The equation of the line through $(a,0)$ and $(0,b)$ is $\frac{x}{a} + \frac{y}{b} = 1$. Since $(4,3)$ is on the line, we have $\frac{4}{a} + \frac{3}{b} = 1$. From this equation we get $\frac{4b + 3a}{ab} = 1$ and $4b + 3a = ab$ then $ab - 4b - 3a = 0$.
Multiplying $(a-4)(b-3)$ gives $ab - 4b - 3a + 12$. Since $ab - 4b - 3a = 0$, we have $(a-4)(b-3) = 12$

Round 7

Find the roots of the quadratic using the quadratic formula.

Adding the roots gives: $\tan\left(\frac{P}{2}\right) + \tan\left(\frac{Q}{2}\right) = -\frac{b}{a}$

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Multiplying the roots gives: $\tan\left(\frac{P}{2}\right)\cdot\tan\left(\frac{Q}{2}\right)=\frac{c}{a}$

Using the addition formula gives: $\tan\left(\frac{P}{2}+\frac{Q}{2}\right)=\frac{\tan\left(\frac{P}{2}\right)+\tan\left(\frac{Q}{2}\right)}{1-\tan\left(\frac{P}{2}\right)\tan\left(\frac{Q}{2}\right)}=\frac{-\frac{b}{a}}{1-\frac{c}{a}}=\frac{-\frac{b}{a}}{\frac{a-c}{a}}=\frac{b}{c-a}$.

So, $\tan\left(\frac{P+Q}{2}\right)=\frac{b}{c-a}$. Since $R=\frac{\pi}{2}$, $P+Q=\frac{\pi}{2}$ which implies $\frac{P+Q}{2}=\frac{\pi}{4}$.

So, $\tan\left(\frac{\pi}{4}\right)=\frac{b}{c-a}$ which gives $1=\frac{b}{c-a}$ and $c-a=b$ and $a+b=c$.

Round 8

Let *y* be the speed of the current and *x* be Jane's paddling speed. The distance that Jane travelled after turning around is $D=1\,mi+(x-y)(1\,hr)$ which is equal to $(x+y)t$, where *t* is the time it took to travel that distance with the stream. The log is carried by the current and it travels 1 mile in time $t+1\,hr$. That equation is $y(t+1\,hr)=1\,mi$. Dropping units (speed in *mph*) we have $1+x-y=xt+yt$ and $yt+y=1$. Substitute the second equation into the first and simplify and you have $x=xt$ and $0=x(t-1)$. So the time *t* is one *hr*. The second equation then shows that the speed of the current is 0.5 *mph*.

Round 9

$$\begin{aligned} Denominator &= \frac{1}{99}+\frac{2}{98}+\cdots+\frac{99}{1}=\frac{100-99}{99}+\frac{100-98}{98}+\cdots+\frac{100-1}{1} \\ &= \frac{100}{99}+\frac{100}{98}+\cdots+\frac{100}{1}-99 \\ &= \frac{100}{99}+\frac{100}{98}+\cdots+\frac{100}{2}+1 \\ &= \frac{100}{99}+\frac{100}{98}+\cdots+\frac{100}{2}+\frac{100}{100} \\ &= 100\left[\frac{1}{99}+\frac{1}{98}+\cdots+\frac{1}{2}+\frac{1}{100}\right] \\ &= 100\cdot Numerator \end{aligned}$$

Hence $A = \frac{1}{100}$.

Round 10

Since $ABCDE < 25000$ either $A = 1$ or $A = 2$ *and* $B = 1, 2,$ *or* 4.
Since $EDCBA$ is also even, we have $A = 2$. We know that $ABCDE$ is even so *E* must be divisible by 2 (either 2, 4, 6, or 8), but 2 has already been used and none of the digits is 6, so this leaves 4 or 8.
Since $4(2BCDE) = EDCB2$,$4E$ must end in 2 so $E = 8$.
From above we know that since $A = 2$ then $B = 1, 2,$ *or* 4, but 2 has already been used so *B* must be either 1 or 4. Also, the only remaining possible digits are 1, 4, 5, 7, and 9.
The following cases must be considered for *B* and *D*:

$B = 1 \quad D = 4$

$B = 1 \quad D = 5$

$B = 1 \quad D = 7$

$B = 1 \quad D = 9$

$B = 4 \quad D = 1$

$B = 4 \quad D = 5$

$B = 4 \quad D = 7$

$B = 4 \quad D = 9$

We have $4(2BCD8) = 8DCB2$ (with 3 over). Therefore, $4(BCD) + 3 = DCB$. Trying the combinations above, $B = 1$ *and* $D = 7$ is the only one that works (i.e. 4*D + 3 = 4*7 + 3 = 31).
Now we have $4(21C78) = 87C12$ and *C* must be either 4, 5, or 9. Trying these for *C*, $C = 9$.