University of North Georgia

Mathematics Tournament

April 2, 2016

Solutions for the Afternoon Team Competition

Round 1

Set up the proportion $\frac{10}{3}=\frac{x}{321{,}000{,}000}$ where x is the area needed for the US population. Solving gives 1,070,000,000 square feet. Convert square feet to square miles giving 38.3809687787 square miles. To get the length of one side of the square, take the square root giving 6.195 miles.

Round 2

100% doesn't change the size, so we must use 80% and 150 % copies. $80\%=\frac{4}{5}$ and $150\% = \frac{3}{2}$

We need the smallest x and y such that $\left(\frac{4}{5}\right)^x\left(\frac{3}{2}\right)^y=\frac{324}{100}$. Simplifying gives $\frac{4^x3^y}{5^x2^y}=\frac{81}{25}$. $x=2,\ y=4$ works and there is no smaller combination.

Round 3

You may notice that between every time when hands are in the same position, they make 90 degrees twice. In 24 hours they will be in the same position 22 times, not counting the original time, so the answer is 44.

Round 4

The area is $A=\pi R^2-\pi r^2=\pi\left(R^2-r^2\right)$. We also have $R^2=4^2+r^2$, so $R^2-r^2=4^2$. So $A=\pi\left(R^2-r^2\right)=\pi\cdot 4^2=16\pi\approx 50.265$

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Round 5

Let the sides of the original rectangle be a and b with $a > b$. Then $a = b + 3$ and $a^2 = 2a(a-3)$. Solving gives $a = 0$ or $a = 6$, but since $a > b > 0$, then $a = 6$. Hence $b = 3$. So the area is 6 * 3 = 18.

Round 6

We rewrite the equation as $(10^x)^2 - 28 \cdot 10^x + 209 - P = 0$. The quadratic equation has exactly one real solution when the discriminant = 0. Setting the discriminant = 0 gives $28^2 - 4(1)(209 - P) = 0$. Simplifying gives $14^2 - (209 - P) = 0$ and $196 - 209 + P = 0$ and $P = 13$.

Then according to the quadratic formula $10^x = 14 \pm \sqrt{14^2 - (209 - 13)} = 14 \pm \sqrt{196 - 196} = 14$, which gives a positive solution for 10^x.

Round 7

The square of the distance between the point $(6, 5)$ and a point $(x, 3x+8)$ on the line $y = 3x + 8$ is $D^2 = (x-6)^2 + (3x+8-5)^2$. Simplifying gives $D^2 = (x-6)^2 + (3x+3)^2$,then $D^2 = x^2 - 12x + 36 + 9x^2 + 18x + 9$ and $D^2 = 10x^2 + 6x + 45$. The last is a quadratic expression with a minimum at its vertex, so the x-value is given by $x = \frac{-6}{2(10)} = -\frac{3}{10} = -0.3$. Substituting into the equation $D^2 = 10x^2 + 6x + 45$ gives the square of the distance as $D^2 = 10(-0.3)^2 + 6(-0.3) + 45 = 44.1$. Taking square roots gives $D = 6.641$.

Round 8

If $a^2 + 5a - 2 = 0$, then $a^2 = -5a + 2$.

So $a^2 + \frac{10}{a} = (-5a+2) + \frac{10}{a} = \frac{-5a^2 + 2a + 10}{a} = \frac{-5(-5a+2) + 2a + 10}{a} = \frac{25a - 10 + 2a + 10}{a} = 27$.

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Round 9

Taking into account the property of the logarithm $\log_a b = \frac{\log_b b}{\log_b a} = \frac{1}{\log_b a}$, the equation can be rewritten as $2\log_{18(x-1)^2(x-7)}\left(x^2-4x+3\right) + \frac{1}{\log_{18(x-1)^2(x-7)}\left(x^2-4x+3\right)} = 3$

Letting $c = \log_{18(x-1)^2(x-7)}\left(x^2-4x+3\right)$ and substituting gives $2c+\frac{1}{c}=3$ and then the quadratic equation $2c^2-3c+1=0$. Solving the quadratic gives the 2 solutions $c=1$ and $c=\frac{1}{2}$.
Now we have 2 cases. Case 1: $c=1$ Recalling that $c=\log_{18(x-1)^2(x-7)}\left(x^2-4x+3\right)$, we have $\log_{18(x-1)^2(x-7)}\left(x^2-4x+3\right)=1$. Solving this logarithmic equation, we confirm that there are no integer solutions for this case. Case 2: $c=\frac{1}{2}$ We have $\log_{18(x-1)^2(x-7)}\left(x^2-4x+3\right)=\frac{1}{2}$. We can rewrite as $x^2-4x+3=\sqrt{18}(x-1)\sqrt{x-7}$. Factoring gives $(x-3)(x-1)=\sqrt{18}(x-1)\sqrt{x-7}$, but $x\neq 1$, so $x=1$ is not a solution. Now we have $(x-3)=\sqrt{18}\sqrt{x-7}$. Squaring both sides gives $x^2-6x+9=18(x-7)$ and then $x^2-24x+135=0$ which gives the two integer solutions $x=15$ and $x=9$.

Round 10

Let $5+\cfrac{6}{5+\cfrac{6}{\vdots}}=x$, then $5+\frac{6}{x}=x$ and $x^2-5x-6=0$. Factoring gives $(x-6)(x-1)=0$. This gives solutions $x=6$ and $x=-1$, but $x\neq -1$. Similarly, let $16-\cfrac{64}{16-\cfrac{64}{\vdots}}=x$, then $16-\frac{64}{x}=x$ and $x^2-16x+64=0$. Factoring gives $(x-8)(x-8)=0$. This gives the solution $x=8$. Thus,

$$\cfrac{6}{5+\cfrac{6}{5+\cfrac{6}{\vdots}}} + \cfrac{8}{16-\cfrac{64}{16-\cfrac{64}{\vdots}}} = \frac{6}{6}+\frac{8}{8}=2.$$

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