Twenty Second Annual University of North Georgia Mathematics Tournament

You may write in this test booklet. Only the electronic form will be graded. Correct answers are awarded one point. Incorrect or blank answers are awarded 0 points.

1. Find the integral: $\int_0^{\frac{\pi}{2}} \sqrt{1+\sin x}\ dx$

   a) 4

   b) $\sqrt{\pi-1}$

   c) 2

   d) $\frac{\sqrt{\pi}}{2}$

   e) None of the above

2. A vehicle enters a 100 mile stretch of an interstate starting at speed equal to zero and stops at the end of it exactly 1.5 hours later. How many times during the drive was the vehicle speed exactly 50 mph?

   a) At least twice

   b) At least three times

   c) At least four times

   d) At least five times

   e) None of the above

3. Find the maximum value of $y = 6\cos x + 14x - 5$ on $\left[-\frac{3\pi}{2}, 0\right]$.

a) 1

b) 3

c) 5

d) 7

e) None of the above

4. If g is the inverse function of $f(x) = 2x + \ln x$, find $g'(2)$.

a) $\frac{1}{3}$

b) $2 + e^2$

c) $\frac{3}{2}$

d) $\frac{1}{2}$

e) None of the above

5. Find the definite integral: $\int_0^1 \sqrt{x - x^2}\, dx$

a) 4

b) $\frac{1}{4}$

c) π

d) $\frac{\pi}{8}$

e) None of the above

6. If f is continuous, then $\int_0^1 f(1-x)\,dx$ is equal to

   a) $-\int_0^1 f(x)\,dx$

   b) $\int_0^1 f(x)\,dx$

   c) $\int_0^1 f(-x)\,dx$

   d) $-\int_0^1 f(-x)\,dx$

   e) None of the above

7. Find $f(t)$ so that $\int f(t)\,dt = \frac{t}{2}\sqrt{4-t^2} + 2\arcsin\left(\frac{t}{2}\right)$.

   a) $\frac{1}{\sqrt{1-\frac{t^2}{4}}} + C$

   b) $\sqrt{4-t^2} + C$

   c) $\arcsin\left(\frac{t}{2}\right) + C$

   d) $t\sqrt{4-t^2} + C$

   e) None of the above

8. Find the number b such that the line $y = b$ divides the region bounded by the curves $y = x^2$ and $y = 4$ into two regions with equal area.

   a) $4^{2/3}$

   b) $4^{3/2}$

   c) $4^{1/3}$

   d) $4^{1/2}$

   e) None of the above

9. Find the definite integral: $\int_0^1 \arcsin x \, dx$.

a) 1

b) $\frac{\pi}{2} - 1$

c) $\frac{\pi}{2}$

d) $\frac{\pi}{3}$

e) None of the above

10. Suppose that $\int_0^x f(t)\,dt = \sin x$, find $f(\pi)$.

a) 1

b) -1

c) 0

d) $\frac{\sqrt{3}}{2}$

e) None of the above

11. Evaluate the limit: $\lim_{x\to\infty} \frac{\frac{\pi}{2} - \arctan x}{\ln\left(\frac{2x-1}{1+2x}\right)}$.

a) 2

b) Does not exist

c) ∞

d) -1

e) None of the above

12. A light house is located on a small island, 3 km away from the nearest point *P* on a straight shoreline, and its light makes four revolutions per minute. How fast is the beam light moving along the shoreline when it is 1 km from *P*?

a) 1600π km/h
b) 480π km/h
c) 1800π km/h
d) 7200π km/h
e) None of the above

13. If f is continuous and $\int_0^2 f(x)\,dx = 6$, evaluate $\int_0^{\frac{\pi}{2}} f(2\sin\theta)\cos\theta\,d\theta$.

a) 3
b) 6
c) 2
d) 12
e) None of the above

14. Assume that f'' is continuous and that $f(1) = 3,\ f'(1) = 2,$ and $\int_0^1 f(x)\,dx = 5.$

Find $\int_0^1 x^2 f''(x)\,dx.$

a) 6
b) 1
c) $\frac{1}{2}$
d) -2
e) None of the above

15. Evaluate the limit: $\lim_{x \to 2^-} \dfrac{|x-2|}{x-2}$

a) -2

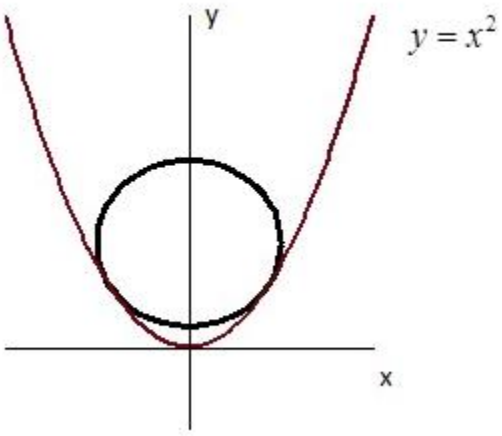
b) -1

c) -4

d) 2

e) None of the above

16. The figure shows a circle with radius 1 inscribed in the parabola $y = x^2$. Find the center of the circle.



a) $\left(0, \dfrac{\sqrt{3}}{2}\right)$

b) $\left(0, \dfrac{5}{4}\right)$

c) $\left(0, \dfrac{3}{4}\right)$

d) $\left(0, \dfrac{\sqrt{3}}{4}\right)$

e) None of the above

17. Evaluate the integral: $\int \frac{dx}{1+e^x}$

a) $x-1+C$

b) $x-\ln e^x+C$

c) $x+\ln e^x+C$

d) $x-\ln\left(e^x+1\right)+C$

e) None of the above

18. Water is poured into a conical cup at the rate of $\frac{2}{3}$ cubic inches per second. If the cup is 6 inches tall and if the top of the cup has a radius of 2 inches, how fast is the water level rising when the water is 4 inches deep?

a) $\frac{3}{8\pi}$ in/sec

b) $\frac{3\pi}{8}$ in/sec

c) $\frac{1}{4}$ in/sec

d) π in/sec

e) None of the above

19. Which of the following expressions equals to $\frac{d^n}{dx^n}\left(\frac{1}{x}\right)$?

a) 0

b) $(-1)^n \frac{n!}{x^n}$

c) $(-1)^n \frac{(n-1)!}{x^{n+1}}$

d) $(-1)^n \frac{n!}{x^{n+1}}$

e) None of the above

20. Let $f(k)=\frac{d^j}{dx^j}\left(e^{kx}\right)$. Find $f'(k)$.

a) $k^{j-1}e^{kx}\left(k^2+j\right)$

b) $k^{j-1}e^{kx}\left(k+j^2\right)$

c) $k^{j-1}e^{kx}\left(kx+j\right)$

d) $k^{j-1}e^{kx}\left(x+jk\right)$

e) None of the above

21. Find the definite integral: $\int_1^4 \frac{1}{\left(1+\sqrt{x}\right)^2}\cdot\frac{1}{\sqrt{x}}\,dx$

a) $\frac{6}{5}$

b) $\frac{4}{9}$

c) $\frac{3}{2}$

d) $\frac{1}{3}$

e) None of the above

22. The sequence of numbers $\left(\frac{2}{1}\right),\left(\frac{3}{2}\right)^2,\left(\frac{4}{3}\right)^3,\ldots,\left(\frac{101}{100}\right)^{100},\ldots$ gets as close as you want to:

a) ∞

b) The number is not defined.

c) 1

d) e

e) None of the above

23. What is the minimum vertical distance between the parabolas $y = x^2 + 1$ and $y = x - x^2$.

a) $\frac{7}{8}$

b) $\frac{5}{8}$

c) $\frac{9}{8}$

d) $\frac{1}{8}$

e) None of the above

24. If the radius of the circle increases from r_1 to r_2, the average rate of change of the area of the circle with respect to the radius is

a) Greater than $2\pi r_2$

b) Less than $2\pi r_1$

c) Equal to $2\pi \frac{r_1 + r_2}{2}$

d) Equal to $2\pi \frac{r_2 - r_1}{2}$

e) None of the above

25. Find the limit: $\lim_{x \to \infty} \left(\sqrt{x^2 + 4x} - x \right)$

a) 1

b) 2

c) 4

d) Does not exist

e) None of the above

26. Find the value(s) of the constant c that make(s) the function $f(x)=\begin{cases} c^2-x^2 & \text{if} \quad x<2 \\ 2(c-x) & \text{if} \quad x\geq 2 \end{cases}$ continuous on $(-\infty,\infty)$.

a) $-1,-3$
b) $4,-2$
c) 2
d) Does not exist
e) None of the above

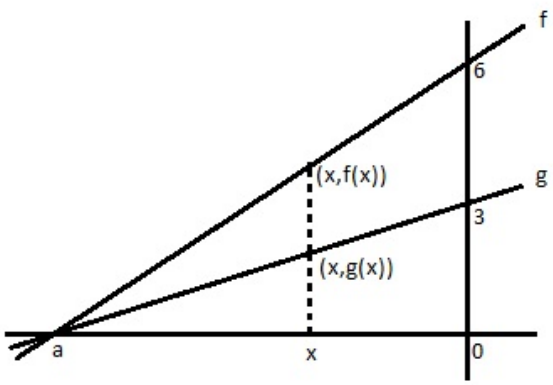
27. If $f(x)=\sqrt{x+\sqrt{x}}$, find $f'(1)$.

a) $\dfrac{3\sqrt{2}}{8}$
b) $\dfrac{\sqrt{2}}{2}$
c) $\dfrac{1}{2}$
d) 1
e) None of the above

28. Find the limit: $\lim\limits_{y\to 0}\dfrac{2e^{2y^2-3y+4}(\sin y\cos y-\sin y)}{y^4-y^2}$

a) 0
b) $2e^4$
c) ∞
d) Does not exist
e) None of the above

29. Suppose you have two linear functions f and g shown below.



Then $\lim_{x \to a} \frac{f(x)}{g(x)}$ is

a) 2

b) Does not exist

c) Not enough information

d) 3

e) None of the above

30. A ball is thrown into the air and its height in feet after t seconds is given by $s(t) = 80t - 16t^2$. It will be at maximum height when its instantaneous velocity is zero. Find its average velocity from the time it is thrown $(t = 0)$ to the time it reaches its maximum height.

a) 50 ft/sec
b) 60 ft/sec
c) 40 ft/sec
d) 32 ft/sec
e) None of the above

31. Suppose that $f(0)=0$ and $f'(0)=2$, and let $g(x)=f\left(-x+f\left(f(x)\right)\right)$.
Find $g'(0)$.

a) 0
b) 2
c) 6
d) 4
e) None of the above

32. Consider the function $f(x)=\begin{cases} x^2 & \text{if } x \text{ is rational, } x\neq 0 \\ -x^2 & \text{if } x \text{ is irrational} \\ \text{undefined} & \text{if } x=0 \end{cases}$.

Then
a) There is no a for which $\lim_{x\to a} f(x)$ exits.
b) Here may be some a for which $\lim_{x\to a} f(x)$ exists, but it is impossible to say without more information.
c) $\lim_{x\to a} f(x)$ exists only when $a=0$.
d) $\lim_{x\to a} f(x)$ exists for infinitely many a.
e) None of the above

33. Find the definite integral: $\int_{1/4}^{1/2} \frac{1}{t^2\sqrt{1-t^2}}\,dt$

a) $\sqrt{3}-\sqrt{15}$
b) $\sqrt{15}-\sqrt{3}$
c) $\sqrt{3}+\sqrt{15}$
d) $-\sqrt{15}-\sqrt{3}$
e) None of the above

34. A plane flying horizontally at an altitude of 1 mile and a speed of 500 mi/h passes directly over a radar station. Find the rate at which the distance from the plane to the station is increasing when it is 2 mi away from the station.

a) $125\sqrt{3}$ mi/h
b) $250\sqrt{2}$ mi/h
c) $250\sqrt{5}$ mi/h
d) $125\sqrt{2}$ mi/h
e) None of the above

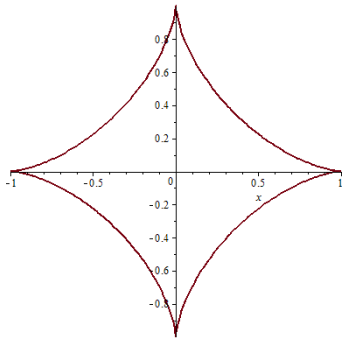
35. Let $f(x) = xe^{-x}$. Find $f^{(2006)}(0)$.

a) -1
b) -2006
c) 2006
d) 1
e) None of the above

36. Suppose $g(x)$ is continuous on $[-1,1]$ with $g(-1) = -1$ and $g(1) = 1$. Which of the following must be true?

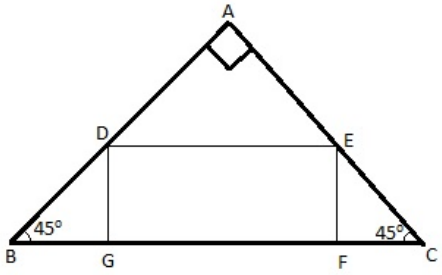
a) There is a value of c in $(-1,1)$ where $g(c)$ equals -1 or 1.
b) There is a unique value of c in $(-1,1)$ where $g(c) = \frac{1}{2}$.
c) There is a value of c in $(-1,1)$ where $g(c)$ equals the area of a circle with radius $\frac{1}{2}$.
d) All of the above
e) None of the above

37. Find the length of the curve with the equation $x^{2/3} + y^{2/3} = 1$



a) 6
b) 4
c) 24
d) 12
e) None of the above

38. In the figure below, find the dimension of the rectangle with maximal area in the $45° - 45° - 90°$ right triangle with legs of length 1.



a) $\sqrt{2}, \sqrt{2}$
b) $\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{4}$.
c) $\frac{1}{2}, \frac{1}{4}$
d) $\frac{1}{6}, 2$
e) None of the above

39. Find the derivative $\frac{df}{dx}$ of the function $f(x) = x^{\sin x}$.

a) $\sin x \cdot x^{\sin x - 1}$

b) $\sin x \cdot x^{\cos x}$

c) $x^{\sin x}\left[\cos x \ln x + \frac{\sin x}{x}\right]$

d) $x^{\sin x} \ln(\sin x)$

e) None of the above

40. Suppose $f(x)$ is differentiable everywhere and $f(x) + 2f(-x) = \sin x$ for all real x. What is the value of $f'\left(\frac{\pi}{4}\right)$?

a) 1

b) -1

c) $\frac{\sqrt{2}}{2}$

d) $-\frac{\sqrt{2}}{2}$

e) None of the above