Twenty-Fourth Annual University of North Georgia Mathematics Tournament April 7, 2018

Morning Component

Good morning!

Please do NOT open this booklet until given the signal to begin.

There are 40 multiple choice questions. Answer the questions on the electronic grading form by giving the best answer to each question.

The scoring will be done by giving one point for each question answered correctly and zero points for each question answered incorrectly or left blank. Thus, it is to your advantage to answer as many questions as possible, even if you have to guess. If there is a tie, question number 8 will be used again as a tie-breaker.

This test was designed to be a CHALLENGE. It is difficult, and you may not have time to complete all questions. Do not worry if you are unable to answer several of the questions. Instead, we hope that you will obtain satisfaction from those questions which you ARE able to answer.

You may write in the test booklet. You may keep your test booklet and any of your scrap papers. Only the electronic grading form will be collected and graded.

Good luck!

Do Not Open Until Signaled.

Twenty Fourth Annual University of North Georgia Mathematics Tournament

You may write in this test booklet. Only the electronic form will be graded. Correct answers are awarded one point. Incorrect or blank answers are awarded 0 points.

1. Find the limit: $\lim_{x\to 1}\left((2-x)^{\tan\left(\frac{\pi x}{2}\right)}\right)$.

   a) $\frac{2}{\pi}$

   b) $\frac{4}{\pi}$

   c) $e^{\left(\frac{4}{\pi}\right)}$

   d) $e^{\left(\frac{2}{\pi}\right)}$

   e) None of the above

2. Evaluate the definite integral: $\int_{\frac{1}{e}}^{e} |\ln(x)|\, dx$.

   a) $e+\frac{2}{e}$

   b) $e-\frac{2}{e}$

   c) $2+\frac{2}{e}$

   d) $2-\frac{2}{e}$

   e) None of the above

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3. If $u = f \circ g \circ h$, $f'(0) = \frac{1}{2}$, $g(-2) = 0$, $g'(-2) = -\frac{1}{2}$, $h(2) = -2$, and $h'(2) = 2$, then $u'(2) = ?$

   a) $-\frac{1}{2}$

   b) $\frac{1}{2}$

   c) -2

   d) 2

   e) None of the above

4. Assume that the function f is differentiable at $a \neq 0$, $f(a) = 0$, and $f'(a) = \frac{1}{a}$.

   Find the limit: $\lim_{x \to a} \frac{x \cdot f(a) - a \cdot f(x)}{x - a}$.

   a) Does not exist

   b) 1

   c) 0

   d) -1

   e) None of the above

5. The distance between the origin O and the *x*-intercept of the tangent line to the positive branch of the hyperbola $xy = 4$ is increasing 3 units per second. Let B be the *y*-intercept of the tangent line. Find the rate at which the distance OB changes, when the *x*-intercept is at the point (15, 0).

   a) $\frac{16}{75}$ units per second

   b) $\frac{32}{75}$ units per second

   c) $-\frac{16}{75}$ units per second

   d) $-\frac{32}{75}$ units per second

   e) None of the above

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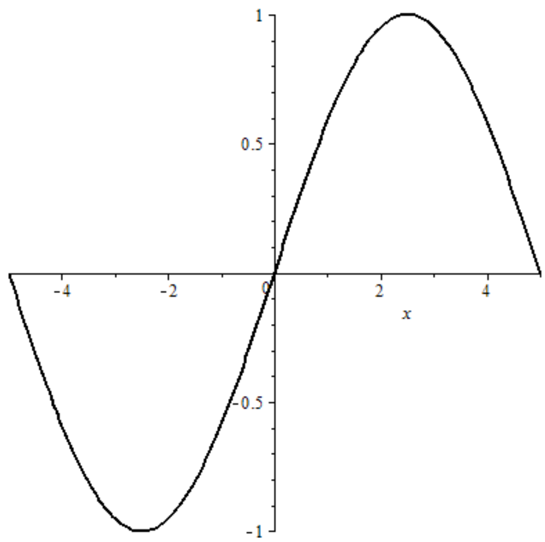
6. The graph of the derivative of the function f , continuous on the interval [-5, 5], is given. What is the correct ordering of the values $f(-5)$, $f(0)$, and $f(5)$?



a) $f(-5) = f(0) = f(5)$

b) $f(0) < f(5) < f(-5)$

c) $f(0) < f(-5) < f(5)$

d) $f(0) < f(-5) = f(5)$

e) None of the above

7. If $f(x+h) - f(x) = 2\sin x \cos h - 2\sin x + 2\cos x \sin h$, find the derivative of the function f.

a) $f'(x) = 2\cos x$

b) $f'(x) = 2\cos x - 1$

c) $f'(x) = 2\sin x$

d) $f'(x) = 2\sin x - 1$

e) None of the above

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Reminder

Question 8 will be used as a tie-breaker, if necessary.

8. The curve $y = \cos^{-1} x$, where $0 \le x \le 1$, is rotated about the *x*-axis. What is the volume generated?

   a) $\pi^2 - 2\pi$

   b) $\pi^2 + 2\pi$

   c) 2

   d) π^2

   e) None of the above

9. Let *C* be the curve $y = (x-1)^3$ and let *L* be the line $3y + x = 0$. Find the equations of all lines that are tangent to *C* and also perpendicular to *L*.

   a) $y = 3x - 6,\ y = 3x - 2$

   b) $y = 3x - 5,\ y = 3x - 1$

   c) $y = 3x + \frac{1}{3},\ y = 3x - \frac{1}{3}$

   d) $y = 3x - 3,\ y = 3x + 3$

   e) None of the above

10. The equation $x^{13} + 7x^3 - 5 = 0$ has

    a) exactly one real root
    b) more than one real root
    c) exactly thirteen real roots
    d) no real root

    e) None of the above

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11. Find the maximum value of the function $f(x) = \dfrac{\sin x + 1}{\sin^2 x + \sin x + 1}$.

a) The maximum value of the function f is $\dfrac{2}{3}$.

b) The maximum value of the function f is 0.

c) The maximum value of the function f is 1.

d) The maximum value of the function f does not exist.

e) None of the above.

12. Find the limit: $\lim\limits_{x\to\infty} x\left(\ln\left(1+\dfrac{x}{2}\right) - \ln\left(\dfrac{x}{2}\right)\right)$.

a) Does not exist
b) 0
c) 1
d) 2
e) None of the above

13. A tangent line to the graph of $y = x^3$ at a point (x_0, y_0) has been drawn. What is (x_0, y_0) if the tangent line passes through the point $(2,4)$ as well?

a) $(1,2)$

b) $(1+\sqrt{3}, 10)$

c) $(1, 10+6\sqrt{3})$

d) $(1-\sqrt{3}, 8)$

e) None of the above

14. Evaluate the following limit by first recognizing the sum as a Riemann sum for a function defined on [0, 1]: $\lim_{x\to\infty}\left(\frac{n}{n^2+1}+\frac{n}{n^2+2^2}+\dots+\frac{n}{n^2+n^2}\right)$.

a) $\frac{\pi}{2}$

b) $\frac{\pi}{3}$

c) $\frac{\pi}{4}$

d) π^2

e) None of the above

15. If $f(1)=10$ and $f'(x)\geq 2$ for $1\leq x\leq 4$, how small can $f(4)$ possibly be?

a) 8

b) 10

c) 14

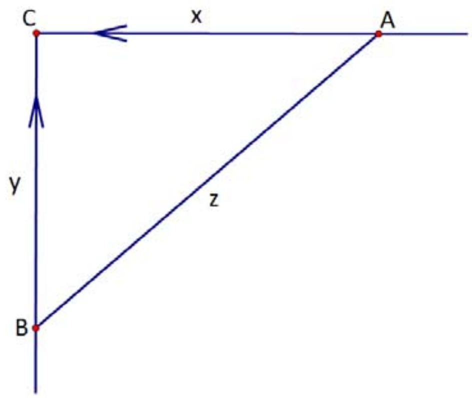
d) 16

e) None of the above

16. A well is 100 feet deep. A bucket weighing 3 pounds has a volume of 2 cubic feet. The bucket is filled with water at the bottom of the well and is then raised at a constant rate of 5 feet per second to the top. Neglecting the weight of the rope, find the work done in raising the bucket if it is discovered that the water is leaking out at a constant rate of 0.01 cubic feet per second. (A cubic foot of water weighs 62.4 pounds.)

a) 10,284 ft-lb
b) 12,156 ft-lb
c) 11,532 ft-lb
d) 11,856 ft-lb
e) None of the above

17. Car A travels west at 50 mi/h and car B travels north at 60 mi/h (See picture below). Both are heading for the intersection of the two roads. At what rate are the cars approaching each other when car A is 0.3 mi and car B is 0.4 mi from the intersection?



a) 78 mi/h
b) 80 mi/h
c) 50 mi/h
d) 67 mi/h
e) None of the above

18. Evaluate the definite integral: $\int_0^1 \frac{(7x-1)^{2018}}{(2x+1)^{2020}}\,dx.$

a) $\frac{1}{10,095}\left(2^{2019}+1\right)$

b) $\frac{1}{18,171}\left(2^{2019}-1\right)$

c) $\frac{2^{2019}}{10,095}$

d) $\frac{7}{2}$

e) None of the above

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19. Let f be a continuous function. Find $f(4)$ if $\int_0^{x^2} f(t)\,dt = x\sin(\pi x)$ for all x.

a) $\dfrac{\pi}{4}$

b) $\dfrac{\pi}{3}$

c) $\dfrac{\pi}{2}$

d) π

e) None of the above

20. Find the limit: $\displaystyle\lim_{x\to 0} \frac{\cos\left(\frac{\pi}{2}\cos x\right)}{\sin(\sin x)}$.

a) Does not exist

b) 0

c) 1

d) $\dfrac{\pi}{2}$

e) None of the above

21. Find minimum value of the function $y = (19 - x)e^{19-x}$.

a) $\dfrac{19}{e^{19}}$

b) $-\dfrac{1}{e}$

c) $-\dfrac{2}{e^2}$

d) No minimum

e) None of the above

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22. If $K(r)=\frac{1}{2}\sin(-2r)$, find $K^{(51)}\left(-\frac{\pi}{8}\right)$.

a) $2^{50}\sqrt{2}$

b) $-2^{50}\sqrt{2}$

c) $2^{49}\sqrt{2}$

d) $-2^{49}\sqrt{2}$

e) None of the above

23. Find the limit: $\lim\limits_{x\to 0^+}\left[\left(x^2+x\right)^{\frac{1}{3}}\cdot\sin\left(\frac{1}{x^2}\right)\right]$.

a) 0

b) 1

c) $\frac{2}{3}$

d) Does not exist

e) None of the above

24. Find the integral: $\int\frac{dx}{(ax+1)(x^2+1)}$, $a>0$.

a) $\frac{1}{a^2+1}\left(a\ln|ax+1|-\frac{a}{2}\ln(x^2+1)+\arctan(x)\right)+C$

b) $\frac{1}{a}\ln|ax+1|-\arctan(x)+C$

c) $-(a^2+1)\arctan(x)-\frac{a}{2}\ln\left(x^2+1\right)+C$

d) $\frac{a}{2}\ln\left(x^2+1\right)+C$

e) None of the above

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25. Find the area of the surface of revolution generated by revolving $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$, for $a > 0$, about the x-axis.

a) $\dfrac{6\pi a^2}{5}$

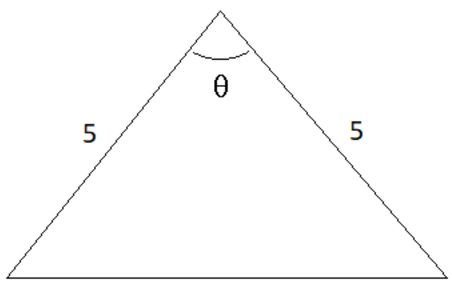
b) $\dfrac{12\pi a^2}{5}$

c) $\dfrac{4\pi a^2}{5}$

d) $\dfrac{8\pi a^2}{5}$

e) None of the above

26. What angle θ between two edges of length 5 will result in an isosceles triangle with the largest area?



a) $\dfrac{4\pi}{7}$

b) $\dfrac{9\pi}{20}$

c) $\dfrac{\pi}{2}$

d) $\dfrac{\pi}{3}$

e) None of the above

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27. Let $f(x)=\begin{cases} \dfrac{\cos x}{\frac{\pi}{2}-x} & if \quad x \neq \dfrac{\pi}{2} \\ \\ a^2+a+1 & if \quad x=\dfrac{\pi}{2} \end{cases}$

For which value of a is the function f continuous at $\frac{\pi}{2}$?

a) $a=-1$

b) $a=2$

c) $a=0$ and $a=-1$

d) $a=-2$

e) None of the above

28. Evaluate the definite integral: $\int_0^{2a} \frac{dx}{(x-a)^2}$

a) $\frac{2}{a}$

b) $-\frac{2}{a}$

c) $\frac{a}{2}$

d) $-\frac{a}{2}$

e) None of the above

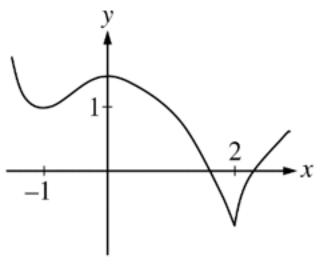
29. Let f be a continuous function defined for all real numbers having the following properties.

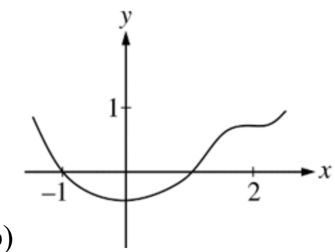
$f'(0) = 0$

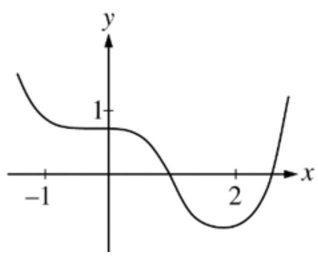
$f''(-1) > 0$

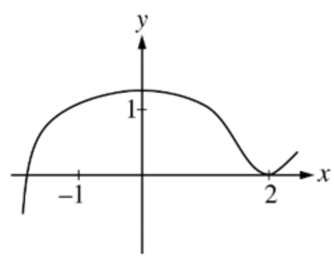
$f''(x) < 0$ if $0 < x < 2$

Which of the following could be a part of the graph of f ?

a) 

b) 

c) 

d) 

e) None of the above

30. Find a monic polynomial, P, of degree 3 such that the graph of P has a local (or relative) maximum at $(x, y) = (-3, 10)$ and a point of inflection when $x = -\frac{5}{3}$.

a) $P(x) = x^3 - 2x + 3$

b) $P(x) = x^3 - 3x^2 + 3x + 1$

c) $P(x) = x^3 + 5x^2 + 3x + 1$

d) $P(x) = x^3 - 5x^2 - 6x + 1$

e) None of the above

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31. Evaluate the definite integral: $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(\cos x + \frac{\sin^5 x}{1+x^2} \right) dx$.

a) 2
b) 0
c) $\frac{\sqrt{3}}{3}$
d) $\frac{\sqrt{2}+1}{2}$
e) None of the above

32. If $g(x) = x + e^x$, what is the value of the derivative of g^{-1} at $g(0)$?

a) $-\frac{1}{2}$
b) 1
c) 0
d) $\frac{1}{2}$
e) None of the above

33. Let f be a positive, continuous function on $[0, a]$. Find

$$\int_0^a \frac{f(x)}{f(x) + f(a-x)} dx.$$

a) a
b) $\frac{a}{2}$
c) $\frac{a}{3}$
d) 0
e) None of the above

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34. If $\tan(xy) = xy$, find $\frac{dy}{dx}$.

a) $-\frac{x}{y}$
b) $\frac{x}{y}$
c) $-\frac{y}{x}$
d) $\frac{y}{x}$
e) None of the above

35. Given $F(x) = x^2$ is an antiderivative of the function $f(x)e^{2x}$. Find the antiderivative of the function $f'(x)e^{2x}$.

a) $-x^2 + 2x + C$
b) $-x^2 + x + C$
c) $2x^2 - 2x + C$
d) $-2x^2 + 2x + C$
e) None of the above

36. Find the limit: $$\lim_{h \to 0} \frac{\sin^7\left(\frac{\pi}{6} + \frac{h}{2}\right) - \left(\frac{1}{2}\right)^7}{h}$$

a) $\frac{7 \cdot \sqrt{3}}{2^8}$
b) $\frac{7 \cdot 3^3}{2^7}$
c) $\frac{7}{2^7}$
d) $\frac{7 \cdot \sqrt{3}}{2^7}$
e) None of the above

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37. Evaluate the definite integral: $\int_1^e \frac{\ln x}{\sqrt{x}}\, dx$.

a) $4+\sqrt{e}$
b) $4-e$
c) $2+e$
d) $4-2\sqrt{e}$
e) None of the above

38. Find the limit: $\lim_{x\to\infty} \frac{\int_0^{x^2} \frac{t^4}{1+t^3}\, dt}{x^4}$.

a) 1
b) $\frac{1}{2}$
c) $\frac{1}{3}$
d) $\frac{1}{4}$
e) None of the above

39. Evaluate the definite integral: $\int_0^\infty \frac{1}{1+e^x}\, dx$.

a) e
b) e^2
c) $\ln 2$
d) $\ln 3$
e) None of the above

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40. If f is continuous on $(-\infty,\infty)$ and $f(x)+f(-x)=\cos^2 x$, then $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} f(x)\,dx =$

a) $\frac{\pi}{2}$
b) π
c) $\frac{\pi}{6}$
d) $\frac{\pi}{4}$
e) None of the above