



***Twenty-Fifth Annual
University of North Georgia
Mathematics Tournament
April 6, 2019***

Morning Component

Good morning!

Please do NOT open this booklet until given the signal to begin.

There are 40 multiple choice questions. Answer the questions on the electronic grading form by giving the best answer to each question.

The scoring will be done by giving one point for each question answered correctly and zero points for each question answered incorrectly or left blank. Thus, it is to your advantage to answer as many questions as possible, even if you have to guess. If there is a tie, question number 28 will be used again as a tie-breaker.

This test was designed to be a CHALLENGE. It is difficult, and you may not have time to complete all questions. Do not worry if you are unable to answer several of the questions. Instead, we hope that you will obtain satisfaction from those questions which you ARE able to answer.

You may write in the test booklet. You may keep your test booklet and any of your scrap papers. Only the electronic grading form will be collected and graded.

Good luck!

Do Not Open Until Signaled.

***Twenty Fifth Annual University of North Georgia
Mathematics Tournament***

You may write in this test booklet. Only the electronic form will be graded. Correct answers are awarded one point. Incorrect or blank answers are awarded 0 points.

1. Find the maximum value of the function $y = \frac{26\sqrt{3}}{3} \cos x + \frac{13\sqrt{3}}{3} x - \frac{13\sqrt{3}}{18} \pi + 11$

on the interval $\left[0, \frac{\pi}{2}\right]$.

a) $\frac{13\sqrt{3}}{6} \pi$

b) 12

c) $\frac{13\sqrt{3}}{2} \pi$

d) 24

e) None of the above

2. Find the limit: $\lim_{x \rightarrow \infty} \frac{3x + |1 - 3x|}{1 - 5x}$.

a) $-\frac{6}{5}$

b) $-\frac{3}{5}$

c) $\frac{1}{5}$

d) $-\frac{1}{5}$

e) None of the above

3. Find the integral: $\int \frac{\tan^{-1}x}{1+\frac{1}{x^2}} dx$.

a) $\frac{(\tan^{-1}x)^2}{x} + C$

b) $\frac{2(\tan^{-1}x)^2}{3} + \frac{x^3}{3} + C$

c) $x^2 \tan^{-1}x - \frac{1}{2} \ln|\tan^{-1}x| - \frac{(\tan^{-1}x)^2}{x} + C$

d) $x \tan^{-1}x - \frac{1}{2} \ln(x^2 + 1) - \frac{(\tan^{-1}x)^2}{x} + C$

e) None of the above

4. The radius of a sphere is measured to be 5 mm. If this measurement is correct to within $\frac{1}{20}$ mm, estimate the propagated error in the volume of the sphere. $V = \frac{4}{3} \pi r^3$.

Leave your answer in terms of π .

a) $\frac{9\pi}{2}$

b) $\frac{\pi}{20}$

c) 4π

d) 5π

e) None of the above

5. Find an equation of the tangent line to the graph of $3x^2 + 4y^2 = 2x$ at a point where the tangent line makes an angle θ with respect to the positive x -axis, where $\tan \theta = \frac{1}{2}$ and the point lies in the first quadrant.

- a) $3x - 6y = 1$
- b) $x - 2y = 1$
- c) $3x - 6y = 2$
- d) $3x - 6y = -1$
- e) None of the above

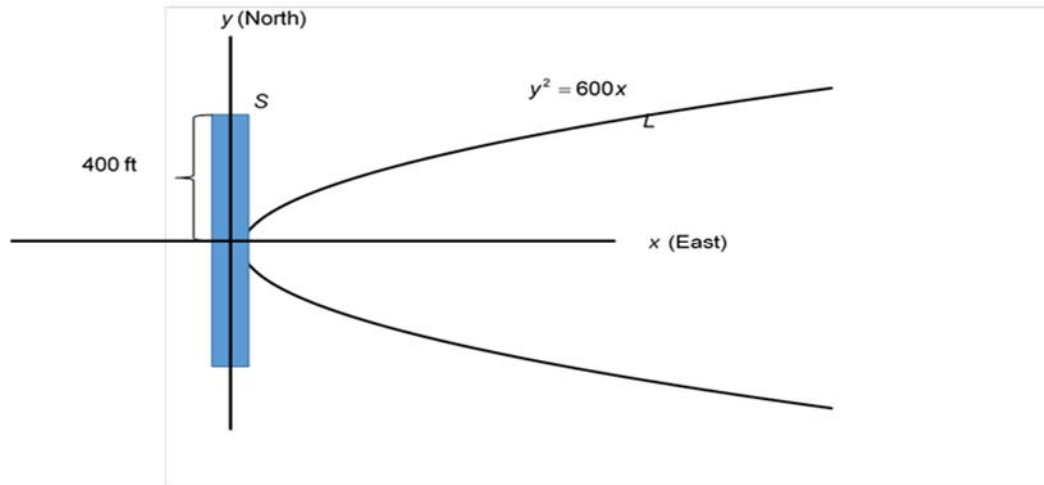
6. The function $f(x) = x^5 + x^3$ has an inverse function g . Find $g'(2)$.

- a) $\frac{1}{6}$
- b) $\frac{1}{7}$
- c) $\frac{1}{8}$
- d) $\frac{1}{9}$
- e) None of the above

7. What is the average area of all circles whose radii are between 1 and 3 meters?

- a) $4\pi \text{ m}^2$
- b) $\frac{13\pi}{3} \text{ m}^2$
- c) $\frac{4\pi}{3} \text{ m}^2$
- d) $\frac{14\pi}{3} \text{ m}^2$
- e) None of the above

8. A railroad train is moving at a speed of 15 mi/h past a station 800 ft long. The track has the shape of the parabola $y^2 = 600x$ as shown. If the sun is just rising in the east, find how fast the shadow S of the locomotive L is moving along the wall at the instant it reaches the end of the wall.



- a) 8.0 mi/h
 b) 8.5 mi/h
 c) 9.0 mi/h
 d) 9.5 mi/h
 e) None of the above
9. Two planes at the same altitude are approaching an airport, one from the north and one from the west. The plane from the north is flying at 250 mph and is 30 mi from the airport. The plane from the west is flying at 200 mph and is 40 mi from the airport. How fast are the planes approaching each other at that instant?
- a) 450 mph
 b) 300 mph
 c) 310 mph
 d) 320 mph
 e) None of the above

10. Find the integral: $\int \sin^2 x \cos^4 x dx$.

a) $\frac{1}{48} \sin^3(2x) + \frac{1}{64} \sin(4x) - \frac{x}{16} + C$

b) $\frac{1}{64} \sin(4x) - \frac{1}{48} \sin^3 x - \frac{x}{16} + C$

c) $\frac{1}{48} \sin(4x) + \frac{1}{64} \sin^3 x - \frac{x}{16} + C$

d) $\frac{1}{16} \sin^3(2x) + \frac{1}{48} \sin(4x) + \frac{x}{16} + C$

e) None of the above

11. Use the Chain Rule and the Second Fundamental Theorem of Calculus to find $\frac{dI}{dx}$,

where $I(x) = \int_{\tan^{-1}(x^2)}^{x^x} \ln t dt$ and $x > 0, x \neq 1$.

a) $x^{x+1} \left((\ln x)^2 + \ln x \right) - \frac{2x}{1+x^4} \ln(\tan^{-1}(x^2))$

b) $x^x \left((\ln x)^2 - \ln x \right) - \frac{x}{1+x^4} \ln(\tan^{-1}(x^2))$

c) $x^{x-1} \left(2 \ln(x-1) + \ln x \right) - \frac{2x}{1-x^4} \ln(\tan^{-1}(x^2))$

d) $x^{x+1} \left((\ln x)^2 - 2 \ln x \right) + \frac{2x}{1+x^4} \ln(\tan^{-1}(x^2))$

e) None of the above.

12. Let $\lfloor x \rfloor$ denote the greatest integer less than or equal to x . Evaluate the definite integral:

$$\int_{0.345}^{4.25} (\lfloor x \rfloor + \lfloor x + 0.5 \rfloor) dx.$$

a) 7

b) 16

c) 17

d) $8 - \sqrt{2}$

e) None of the above

13. Find the limit: $\lim_{x \rightarrow \infty} (3^x + 3^{2x})^{\frac{1}{x}}$.

- a) 0
- b) 1
- c) 9
- d) e
- e) None of the above

14. The set of points (x, y) in the plane satisfying $x^{\frac{2}{5}} + |y| = 1$ forms a curve enclosing a region. Compute its area.

- a) $\frac{2}{7}$
- b) $\frac{3}{14}$
- c) $2\sqrt{2}$
- d) $\frac{8}{7}$
- e) None of the above

15. Find the limit: $\lim_{x \rightarrow \infty} \left(\frac{x^2 + x + 3}{x^2 + 3x + 5} \right)^x$.

- a) e^{-2}
- b) $\frac{1}{e}$
- c) 1
- d) $\frac{3}{5}$
- e) None of the above

16. Evaluate the definite integral: $\int_1^2 \frac{1+x^2}{1+x^4} dx$.

- a) $\frac{1}{\sqrt{2}} \ln(\sqrt{2}-1)$
- b) $\sqrt{2} \ln(\sqrt{2}+1)$
- c) $\frac{1}{2}$
- d) $\frac{\sqrt{2}}{2} \arctan\left(\frac{3\sqrt{2}}{4}\right)$
- e) None of the above

17. Evaluate the definite integral: $\int_{-\pi}^{\pi} \sin(\sin(x)) + \cos(x) dx$.

- a) $\cos(\sin(1))$
- b) π
- c) 0
- d) $\sin(\cos(1))$
- e) None of the above

18. Find the limit: $\lim_{x \rightarrow -\infty} \frac{x^3 + \sin(x^2)}{\sqrt{x^6 + 3x^3 + 1}}$.

- a) $-\infty$
- b) 1
- c) -1
- d) 0
- e) None of the above

19. Find the limit: $\lim_{x \rightarrow 1^+} \frac{\int_1^{x^2} \cos(t^4) dt}{x-1}$.

- a) 0
- b) ∞
- c) $\sin(-1)$
- d) $2 \cos(1)$
- e) None of the above

20. Evaluate the definite integral: $\int_{-2}^0 |x^2 - 2x - 3| dx$.

- a) 4
- b) $\frac{2}{3}$
- c) 10
- d) $\frac{10}{3}$
- e) None of the above

21. Let g be the inverse of $f(t) = t + \sin(t)$. Find $g'\left(\frac{\pi}{2} + 1\right)$.

- a) 0
- b) 1
- c) $\frac{\pi}{2}$
- d) $\sin(1)$
- e) None of the above

22. Let f be the continuous function on \mathbb{R} . If $\int_0^{\frac{\pi}{4}} f(\tan x) dx = 4$ and $\int_0^1 \frac{x^2 f(x)}{x^2 + 1} dx = 2$,

what is the value of $\int_0^1 f(x) dx$?

- a) 2
- b) 4
- c) 6
- d) 8
- e) None of the above

23. Let the function f be such that $f(2) = -\frac{1}{25}$ and $f'(x) = 4x^3 [f(x)]^2$ for all $x \in \mathbb{R}$.

What is the value of $f(1)$?

- a) $-\frac{41}{400}$
- b) $-\frac{1}{10}$
- c) $-\frac{391}{400}$
- d) $-\frac{1}{40}$
- e) None of the above

24. Let f be a function and c a constant such that $3x^{-2} = 2 + \int_c^{x^2} \frac{f(t)}{t^3} dt$.

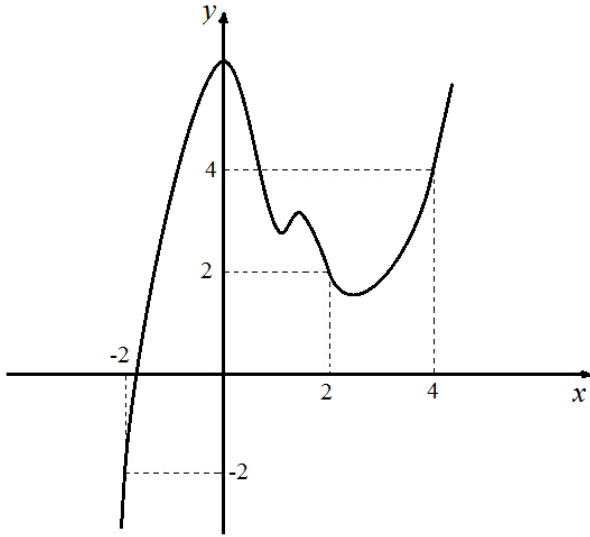
What is the value of $f(c)$?

- a) $\frac{5}{2}$
- b) $\frac{2}{5}$
- c) $-\frac{9}{2}$
- d) -3
- e) None of the above

25. Evaluate the definite integral: $\int_{\sqrt{3}}^{\infty} \frac{dx}{x\sqrt{x^2+1}}$.

- a) $\frac{1}{2} \ln 3$
- b) e^2
- c) ∞
- d) $\ln 3$
- e) None of the above

26. Let f be the function with the graph of f' given by the following:



Let $h(x) = 2f(x) - x^2$. Which of the following is true?

- a) $h(4) = h(-2) > h(2)$
- b) $h(4) = h(-2) < h(2)$
- c) $h(2) > h(4) > h(-2)$
- d) $h(2) > h(-2) > h(4)$
- e) None of the above

27. Use the fact that $(y-x)(y+x)^2 = 1$ to evaluate $\int_0^{\infty} (y-x) dx$.

- a) $\frac{3}{4}$
- b) $\frac{1}{4}$
- c) $\frac{1}{2}$
- d) 1
- e) None of the above

Reminder

Question 28 will be used as a tie-breaker, if necessary.

28. Find the volume of the solid obtained by rotating the region enclosed by $y = \frac{1}{x^2 + 1}$, $x = 1$, $x = 0$, $y = 0$ about the line $x = 0$.
- a) $\frac{\pi}{2} \ln 2$
 - b) $\pi \ln 2$
 - c) 2π
 - d) $2\pi \ln 2$
 - e) None of the above
29. Suppose f is a differentiable function such that $f'(x) \leq 3$ for all $x \in [-3, 4]$. If $f(-3) = 4$, what is the largest possible value of $f(4)$?
- a) 10
 - b) -15
 - c) 25
 - d) 60
 - e) None of the above
30. Suppose that on the interval I , f is positive and concave up. Furthermore, assume that f'' exists and let $g(x) = (f(x))^2$. Which of the following is true for the function g ?
- a) $g(x)$ is concave down on I
 - b) $g(x)$ is concave up on I
 - c) There is not enough information to verify $g(x)$ is concave up or concave down on I
 - d) $g(x)$ has one inflection point in the interval I
 - e) None of the above

31. Let $f(x) = \sqrt{|x|}$, $x \in \mathbb{R}$. Find $f'(0)$.

- a) ∞
- b) $-\infty$
- c) 0
- d) $\frac{1}{2}$
- e) None of the above

32. Find the limit: $\lim_{x \rightarrow 0} (1 + x^2)^{\cot(x)}$.

- a) Does not exist
- b) π
- c) 0
- d) 1
- e) None of the above

33. Find the limit: $\lim_{n \rightarrow \infty} \left(\frac{1}{n+1} + \frac{1}{n+2} + \dots + \frac{1}{n+2n} \right)$.

- a) Does not exist
- b) 0
- c) $\ln 2$
- d) $\ln 3$
- e) None of the above

34. Find the limit: $\lim_{x \rightarrow \infty} x \left(\sqrt{x^2 + 1} - \sqrt[3]{x^3 + 1} \right)$.

- a) ∞
- b) 0
- c) $\frac{1}{2}$
- d) $\frac{3}{4}$
- e) None of the above

35. Evaluate the definite integral: $\int_0^{\pi} \frac{x \sin(x)}{1 + \cos^2(x)} dx$.

- a) 1
- b) $\frac{\pi}{2}$
- c) $\frac{\pi^2}{4}$
- d) π
- e) None of the above

36. Find the maximum value of the function $f(x) = \sin^4(x) + \cos^4(x)$.

- a) 2
- b) $\frac{\sqrt{2}}{2}$
- c) $\sqrt{2}$
- d) $\frac{1}{2}$
- e) None of the above

37. Suppose $f(x+y) = f(x) + f(y) + x^2y + xy^2$ for all real numbers x, y and $\lim_{x \rightarrow 0} \frac{f(x)}{x} = 1$.

Find $f'(-1)$.

- a) 1
- b) 0
- c) -2
- d) 2
- e) None of the above

38. The point on the curve $x^2 + 2y = 0$ that is closest to the point $\left(0, -\frac{1}{2}\right)$ occurs where y is

- a) $\frac{1}{2}$
- b) 0
- c) $-\frac{1}{2}$
- d) -1
- e) None of the above

39. Let $f(x) = \begin{cases} 8 - x^2 & \text{for } -2 \leq x \leq 2 \\ x^2 & \text{elsewhere} \end{cases}$. What is the value of $A = \int_{-1}^3 f(x) dx$?

- a) $0 < A < 8$
- b) $8 < A < 16$
- c) $16 < A < 24$
- d) $24 < A < 32$
- e) None of the above

40. For small values of k , the function $f(x) = \sqrt[4]{16+k}$ is best approximated by which of the following?

- a) $4 + \frac{k}{32}$
- b) $2 + \frac{k}{32}$
- c) $4 - \frac{k}{32}$
- d) $2 - \frac{k}{32}$
- e) None of the above